OXFORD IB DIPLOMA PROGRAMME

TEACHER NOTES

MATHEMATICS:
ANALYSIS AND APPROACHES

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

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1 From patterns to generalizations: sequences, series and proofs

Essential understanding

Number and algebra allow us to represent patterns, show equivalencies and make generalizations which enable us to model real-world situations. Algebra is an abstraction of numerical concepts and employs variables which allow us to solve mathematical problems.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Modelling real-life situations with the structure of arithmetic and geometric sequences and series allows for prediction, analysis and interpretation.
- Different representations of numbers enable equivalent quantities to be compared and used in calculations with ease to an appropriate degree of accuracy.
- Numbers and formulae can appear in different, but equivalent, forms, or representations, which can help us to establish identities.
- Formulas are a generalisation made on the basis of specific examples, which can then be extended to new examples.
- Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns.
- The binomial theorem is a generalization which provides an efficient method for expanding binomial expressions.
- Proof serves to validate mathematical formulae and the equivalence of identities.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
Patterns in numbers inform the development of algebraic expressions and equations that can be applied to find unknowns.	Investigation 2
Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns.	Investigation 3
Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns with speed and accuracy.	Investigation 4
Modelling allows for prediction, analysis, and interpretation.	Investigation 5
The sum of an infinite geometric series tends to a finite number when the individual terms tend to zero.	Investigation 6

Conceptual understandings	Investigation
The common ratio of a geometric series allows us to determine whether the series converges to a finite value or diverges to infinity.	Investigation 7
Modelling allows for prediction, analysis, and interpretation which enable critical decision making.	Investigation 8
The growth model allows you to determine whether an Arithmetic sequence or a Geometric sequence grows faster.	Investigation 9
Proof allows conclusions to be reached more elegantly than by alternative cumbersome ways.	Investigation 13
The binomial theorem uses combinations to calculate the coefficients in the expansion and these coefficients display symmetry about the centre.	Investigation 18
Numbers and formulae can appear in different but equivalent forms or representations to establish identities.	Investigation 19

Syllabus sections covered in this chapter:

- SL1.2*
- SL1.3*
- SL1.4*
- SL1.6
- SL1.8
- SL1.9
- AHL1.10
- AHL1.15

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students’ understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 3: Solving linear equations, algebraic fractions, surds	Page 24: Example 21 Page 39: Example 32 Page 43: Example 37 Page 59: Example 48 Page 64: Example 53	Page 19: Example 15 Page 20: Example 18 Page 25: Example 23 Page 52: Example 43 Page 56: Example 47	Pages 9, 31, 65

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 65	N/A	N/A

1.1 Sequences, series and sigma notation

Investigation 1

1, 121, 12321, 1234321

1 The answers are palindromic numbers, i.e. when written backwards they give the same number. Also the digits represent the first positive integers in order.

2 Yes it will continue up to a certain limit

$11111^2 = 123454321$, $111111^2 = 12345654321$, etc.

3 The pattern would break when you have a string longer than nine 1s. This happens because we are working in base 10, and the middle sum will be bigger than 9 when this happens.

International-mindedness

Where did numbers come from?

Answer: The history of number from Sumerians and its development to the present Arabic system is a fascinating development to trace. You might want to go back to the Ishango bone, evidence of counting from 20 000 years ago.

Investigation 2 – Curious numbers

Conceptual understanding:

Patterns in numbers inform the development of algebraic expressions and equations that can be applied to find unknowns.

1 When there are 9 tiles along the diagonals

$9 = 5 + 4 \Rightarrow 5 \times 5$ square
So when there are 13 tiles along diagonals
$13 = 7 + 6 \Rightarrow 7 \times 7$ square

2 49 tiles

3 $15 = 8 + 7 \Rightarrow 8 \times 8$ square

4 $135 = 68 + 67 \Rightarrow 68 \times 68$ square

5 When the number of tiles along the diagonal is 4 you have a 2×2 square.

You cannot have a square with 6 tiles along a diagonal.

When there are 8 tiles along a diagonal you have a 4×4 square.

No square with 10 tiles along a diagonal is possible.

When there are 12 tiles along a diagonal you have a 6×6 square.

6 Students' own answers.

7 For odd number of tiles along a diagonal:

$2n + 1 = (n + 1) + n \Rightarrow (n + 1)^2$ tiles are required.

For even number of tiles along a diagonal:

$2n = n + n \Rightarrow n^2$ tiles are required.

8 **(This is the conceptual understanding):** Patterns in numbers inform the development of algebraic expressions and equations that can be applied to find unknowns

TOK

Do the names that we give things impact how we understand them?

Answer: For instance, palindromic numbers.

Some large numbers are named, the google and the googolplex, while others are represented in this form.

How important is the language used to describe mathematics?

TOK

Is mathematics a language?

Answer: You will see the use of several alphabets in mathematical notation (e.g., the use of capital sigma for the sum). One point of view is that mathematics is not only a language but is the only language shared by humans around the world. For example, pi is 3.14159… regardless of what culture, language, nationality or religion you have.

A counterclaim might be whether or not we can communicate our ideas without the use of another spoken tongue.

Developing Inquiry skills

Now go back to the opening question. Suppose the length of each side of the first triangle is 81 cm. Can you work out the length of each side of the figure in each iteration? Tabulate your results and try to find a pattern and then make a conjecture.

Answer: To construct a new iteration, each side is divided into three. Therefore, if the side length of the first iteration is 81 cm, then the side length of the second iteration will be 27 cm, the third iteration 9 cm, and the fourth 3 cm.

The sequence follows the formula $81 \times \frac{1}{3}^{u-1}$

1.2 Arithmetic and geometric sequences and series

Investigation 3

Conceptual understanding:

Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns.

1

Number of people ahead of you	Distance of your first tray to machine, d (m)	Waiting time, T (s)
0	0	0
1	1.8	18
2	3.6	36
. . . .	. . . .	. . . .
n	$1.8n$	$18n$

2 Both the second and third column are linear relations.

3

Number of people ahead of you	Distance of your first tray to machine, d (m)	Waiting time, T (s)
0	0.30	3
1	2.4	24
2	4.5	45
. . . .	. . . .	. . . .
n	$2.1n + 0.3$	$21n + 3$

4 The patterns are still linear, however a constant is added; i.e. relations are not directly proportional.

5

	50 cm		**60 cm**		**80 cm**	
Number of people ahead of you	**Distance of your first tray to machine, *d* (m)**	**Waiting time, *T* (s)**	**Distance of your first tray to machine, *d* (m)**	**Waiting time, *T* (s)**	**Distance of your first tray to machine, *d* (m)**	**Waiting time, *T* (s)**
0	0.5	5	0.6	6	0.8	8
1	2.8	28	3	30	3.4	34
2	5.1	51	5.4	54	6	60
. . . .	. . . .	. . . .				
n	$2.3n + 0.5$	$23n + 5$	$2.4n + 0.6$	$24n + 6$	$2.6n + 0.8$	$26n + 8$

6 The patterns are still linear but the coefficient of n and the constant terms have changed.

7 **Factual:** What do you notice about consecutive terms in the second and third columns?

Answer: Consecutive numbers in the second and third columns differ by a constant.

8 **Factual:** How would you generalize the relationship between the distance from the machine to your first tray and the number of people ahead of you?

Answer: The pattern observed gives $d = (3 \times 0.6 + k)n + k$ where k is the distance between trays of individuals.

9 **Factual:** Write down the relationship between the waiting time and the number of people ahead of you.

Answer: Since time = distance ÷ speed, it follows that

$$T = \frac{(3 \times 0.6 + k)n + k}{0.1} = 10\left((3 \times 0.6 + k)n + k\right)$$

10 **Conceptual:** What common patterns generate the relationships developed in this investigation?

Answer: Linear patterns or linear relations.

(this leads to the conceptual understanding): Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns.

Investigation 4

Conceptual understanding:

Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns with speed and accuracy.

1 28th card

2 $56 \times 27 + 28 = 1540$

3 $500 \times 1001 = 500500$

4 **Factual:** Explain the importance of the actual number of terms added.

Answer: If the number of terms is even there is no card left over when pairing, whereas if the number of cards is odd when pairing first and last cards, a middle card is left over.

5 **a** First even number is 2, and 100th even number is 200. There are 50 even numbers that are less than 100.

$50 \times 202 = 10100$

b First multiple is 3 and the last multiple is 999. There are 333 multiples of 3 that are less than 1000.

$(166 \times 1002) + (167 \times 3) = 166833$

6 **Conceptual:** How was Michela's and Grisha's method more efficient?

Answer (this is the conceptual understanding): Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns with speed and accuracy.

Reflect: Why can *n* not be a rational or a negative number?

Answer: There can only be a positive, integer number of terms in a series.

TOK

How is intuition used in mathematics?

Answer: Gauss' method for adding up integers from 1 to 100. You might want to look at inductive and deductive methods of proof.

Is there a body of knowledge called intuitive mathematics? If so, how do these intuitions hinder or facilitate problem solving?

Try questions such as:

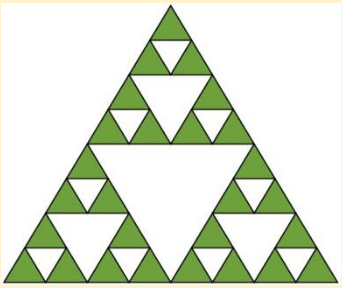
- How many lines pass through any given two points?
- A coin is tossed 12 times. The first 11 all come up heads. What would you expect the next toss to give? Why?
- Which set has more members, the set of rational numbers or the set of irrational numbers?

Investigation 5

Conceptual understanding:

Modelling allows for prediction, analysis, and interpretation

1



2

Stage	0	1	2	3
Number of green triangles	1	3	9	27
Length of one side of one green triangle	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
Area of each green triangle	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$

3 **Factual:** What patterns emerge from each of the three rows of the table?

Answer:

Along the first row you multiply previous term by 3

Along the second row you multiply the previous term by $\frac{1}{2}$

Along the third row you multiply the previous term by $\frac{1}{4}$

4 **Factual:** What do these three patterns have in common?

Answer: To obtain the next term you multiply by a constant number in each case.

5 Number of green triangles in stages 4 to 6 would be stage 4: $27 \times 3 = 81$
stage 5: $81 \times 3 = 243$
stage 6: $243 \times 3 = 729$

Length of side in stages 4 to 6 would be stage 4: $\frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$
stage 5: $\frac{1}{16} \times \frac{1}{2} = \frac{1}{32}$
stage 6: $\frac{1}{32} \times \frac{1}{2} = \frac{1}{64}$

The areas in stages 4 to 6 would be stage 4: $\frac{1}{64} \times \frac{1}{4} = \frac{1}{256}$

stage 5: $\frac{1}{256} \times \frac{1}{4} = \frac{1}{1024}$

stage 6: $\frac{1}{1024} \times \frac{1}{4} = \frac{1}{4096}$

6 **Conceptual:** How would you compare the sets of numbers obtained?

Answer (this is the conceptual understanding): Modelling allows for prediction, analysis, and interpretation.

TOK

Is all knowledge concerned with identification and use of patterns?

Answer: Consider Fibonacci numbers and connections with the Golden ratio.

An opportunity to use a TOK mantra "how do we know what we know?"

Questions might include:

- To what extent do ways of knowing prevent us from deluding ourselves?
- Is a pattern only useful if it simplifies things?
- What is the role of an anomaly in discovery?
- What does it take to know something?
- Is it enough for knowledge to be shared by your teacher or do you need to discover it for yourself?

Investigation 6

Conceptual understanding:

The sum of an infinite geometric series tends to a finite number when the individual terms tend to zero.

1

Line segment	Length of string segment (cm)	Total length of segments (cm)
CD	50	50
DE	25	75
EF	12.5	87.5
FG	6.25	93.75

2 **Factual:** As this process continues indefinitely, what do you notice about the length of the line segments? What about the total length of segments?

Answer: The length of the line segments tend to zero, whilst the total length of the segments tend to 100.

3 **Factual:** What type of sequence is this?

Answer: This is a geometric sequence.

3, 4, 5

n	$\frac{50\times(1-(0.5)^n)}{(1-0.5)}$	$\frac{200}{3}\times\frac{\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\left(\frac{1}{3}\right)}$	$\frac{300}{4}\times\frac{\left(1-\left(\frac{1}{4}\right)^n\right)}{1-\left(\frac{1}{4}\right)}$	$\frac{400}{5}\times\left(\frac{1-\left(\frac{1}{5}\right)^n}{1-\left(\frac{1}{5}\right)}\right)$
1	50	66.66666667	75	80
2	75	88.88888889	93.75	96
3	87.5	96.2962963	98.4375	99.2
4	93.75	98.7654321	99.609375	99.84
5	96.875	99.58847737	99.90234375	99.968
6	98.4375	99.86282579	99.97558594	99.9936
7	99.21875	99.95427526	99.99389648	99.99872
8	99.609375	99.98475842	99.99847412	99.999744
9	99.8046875	99.99491947	99.99961853	99.9999488
10	99.90234375	99.99830649	99.99990463	99.99998976
11	99.95117188	99.9994355	99.99997616	99.99999795
12	99.97558594	99.99981183	99.99999404	99.99999959
13	99.98779297	99.99993728	99.99999851	99.99999992
14	99.99389648	99.99997909	99.99999963	99.99999998
15	99.99694824	99.99999303	99.99999991	100
16	99.99847412	99.99999768	99.99999998	100
17	99.99923706	99.99999923	99.99999999	100
18	99.99961853	99.99999974	100	100
19	99.99980927	99.99999991	100	100
20	99.99990463	99.99999997	100	100
21	99.99995232	99.99999999	100	100
22	99.99997616	100	100	100
23	99.99998808	100	100	100
24	99.99999404	100	100	100
25	99.99999702	100	100	100
26	99.99999851	100	100	100
27	99.99999925	100	100	100
28	99.99999963	100	100	100
29	99.99999981	100	100	100
30	99.99999991	100	100	100

6

Factual: Why were you asked to change the length of the string cut?

Answer: This helped show that the fraction cut determines how fast the total length gets closer to 100 cm.

Conceptual: How has this process help you analyse the situation?

Answer: By modelling for various lengths we could better analyse the situation.

Conceptual: How can the sum of an infinite series converge to a finite number?

Answer (this is the conceptual understanding): The sum of an infinite geometric series tends to a finite number when the individual terms tend to zero.

Investigation 7

Conceptual understanding:

The common ratio of a geometric series allows us to determine whether the series converges to a finite value or diverges to infinity.

1

n	3^n	$(-2)^n$	$(1.5)^n$	$(0.5)^n$	$(-0.2)^n$	$(-0.75)^n$
1	3	-2	1.5	0.5	-0.2	-0.75
2	9	4	2.25	0.25	0.04	0.5625
3	27	-8	3.375	0.125	-0.008	-0.421875
4	81	16	5.0625	0.0625	0.0016	0.316406
5	243	-32	7.59375	0.03125	-0.00032	-0.237305
6	729	64	11.39063	0.015625	0.000064	0.177979
7	2187	-128	17.08594	0.007813	-1.28E-05	-0.133484
8	6561	256	25.62891	0.003906	2.56E-06	0.100113
9	19683	-512	38.44336	0.001953	-5.12E-07	-0.075085
10	59049	1024	57.66504	0.000977	1.02E-07	0.056314
11	177147	-2048	86.49756	0.000488	-2.05E-08	-0.042235
12	531441	4096	129.7463	0.000244	4.1E-09	0.031676
13	1594323	-8192	194.6195	0.000122	-8.19E-10	-0.023757
14	4782969	16384	291.9293	6.1E-05	1.64E-10	0.017818
15	14348907	-32768	437.8939	3.05E-05	-3.28E-11	-0.013363
16	43046721	65536	656.8408	1.53E-05	6.55E-12	0.010023
17	129140163	-131072	985.2613	7.63E-06	-1.31E-12	-0.007517
18	387420489	262144	1477.892	3.81E-06	2.62E-13	0.005638
19	1162261467	-524288	2216.838	1.91E-06	-5.24E-14	-0.004228

2 **Factual:** What is the value of the common ratio?

Answer: The common ratio is the number that is raised to the power n in each case.

3 **Conceptual:** What role does the value of the common ratio play in a geometric series?

Answer (this is the conceptual understanding): The common ratio of a geometric series allows us to determine whether the series converges to a finite value or diverges to infinity. If $|r| < 1$ then the series converges.

4 **a** Consider series for 3^n

b Consider series for $(0.5)^n$

c Consider series for $(-2)^n$

d Consider series for $(-0.2)^n$ or $(-0.75)^n$

e Compare series for $(-0.2)^n$ and $(-0.75)^n$, or consider another sequence $\left(a^n\right)$ where $a < 1$ but a is close to 1.

f When $-1 < r < 1$, $\lim_{n\to\infty}\left(r^n\right) = 0$ so then $S_n = \frac{u_1\left(1-r^n\right)}{1-r} \to \frac{u_1}{1-r}$

Developing inquiry skills

Go back to the original question about Koch's snowflake and try to address the following, assuming that the length of each side of the original triangle is 81 cm:

Calculate the perimeter of the snowflake at each iteration.

Answer: 243 cm, 324 cm, 432 cm, 576 cm

Calculate the area of the snowflake at each iteration.

Answer: 2841 cm^2, 3788 cm^2, 5050.67 cm^2, 6734.22 cm^2

Tabulate the results and explain the number patterns that you observe.

Answer: The perimeter and area increase by $\frac{1}{3}$ with each iteration.

Create a model that helps you to generalize the perimeter and area at any iteration.

Answer: $243 + \left(\frac{4}{3}\right)^{u-1}$; $2841 + \left(\frac{4}{3}\right)^{u-1}$

TOK

How do mathematicians reconcile the fact that some conclusions conflict with intuition?

Answer: Consider for instance that a finite area can be bounded by an infinite perimeter.

Investigation 8

Conceptual understanding:

Modelling allows for prediction, analysis, and interpretation which enable critical decision making.

Answers to questions 1-4 are contained in the table.

Provider B							Provider A				
Number of Notebooks	Number of 3 packs of 100	Number of packs of 100	Cost Price	Cost price per Notebook (cents)	Selling Price (45c)	Profit	Number of 6 packs of 20	Number of packs of 20	Cost Price	Cost price per Notebook (cents)	Profit
500	1	2	192	0.384	225	33	4	1	170	0.34	55
600	2	0	**192**	**0.32**	270	78	5	0	200	**0.333333**	70
700	2	1	240	0.342857	315	75	5	5	250	0.357143	65
800	2	2	288	0.36	360	72	6	4	280	0.35	80
900	3	0	**288**	**0.32**	405	117	7	3	310	0.344444	95
1000	3	1	336	0.336	450	114	8	2	340	0.34	110
1100	3	2	384	0.349091	495	111	9	1	370	0.336364	125
1200	4	0	**384**	**0.32**	540	156	10	0	400	0.333333	140
1300	4	1	432	0.332308	585	153	10	5	450	0.346154	135
1400	4	2	480	0.342857	630	150	11	4	480	0.342857	150
1500	5	0	**480**	**0.32**	675	195	12	3	510	0.34	165
1600	5	1	528	0.33	720	192	13	2	540	0.3375	180
1700	5	2	576	0.338824	765	189	14	1	570	0.335294	195
1800	6	0	**576**	**0.32**	810	234	15	0	600	0.333333	210
1900	6	1	624	0.328421	855	231	15	5	650	0.342105	205
2000	6	2	672	0.336	900	228	16	4	680	0.34	220
2100	7	0	**672**	**0.32**	945	273	17	3	710	0.338095	235
2200	7	1	720	0.327273	990	270	18	2	740	0.336364	250
2300	7	2	768	0.333913	1035	267	19	1	770	0.334783	265
2400	8	0	**768**	**0.32**	1080	312	20	0	800	0.333333	280
2500	8	1	816	0.3264	1125	309	20	5	850	0.34	275
2600	8	2	864	0.332308	1170	306	21	4	880	0.338462	290
2700	9	0	**864**	**0.32**	1215	351	22	3	910	0.337037	305
2800	9	1	912	0.325714	1260	348	23	2	940	0.335714	320
2900	9	2	960	0.331034	1305	345	24	1	970	0.334483	335
3000	10	0	**960**	**0.32**	1350	390	25	0	1000	0.333333	350

The cheapest rate from provider A would be if the stationer orders the following numbers: 600, 1200, 1800, 2400, 3000
The cheapest rate from provider B would be if the stationer orders the following numbers: 600, 900, 1200, 1500, 1800, 2100, ...
The offer from provider A is cheaper if the stationer orders any of the following numbers: 500, 800, 1100, 1400, 1700
This list forms a finite arithmetic sequence whereas the previous ones would be infinite should we go beyond 3000 notebooks
600 notebooks from A and 900 from B would minimise his cost.

Total Cost Price =	*488*			
Total Selling Price =	*675*			
			Provider B	*Provider A*
% Profit	*38.31967*		*40.625*	*32.35294*

5 This work would help the stationer to choose which suppliers to use, and how much product to buy.

6 The selling price is more likely to vary with demand of notebooks; the price may increase at 'back to school' time!

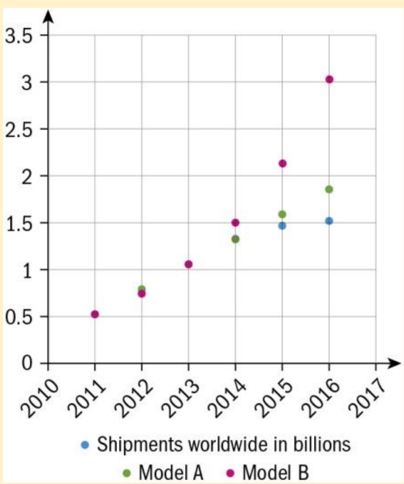
Investigation 9

Conceptual understanding:

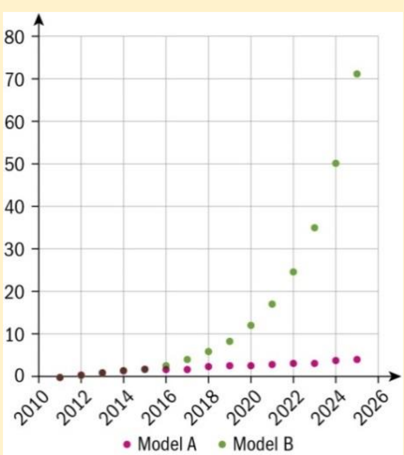
The growth model allows you to determine whether an Arithmetic sequence or a Geometric sequence grows faster.

1-3

Year	Shipments worldwide in billions	d	r				Year	Model A	Model B
2011	0.52						2011	0.52	0.52
2012	0.74	0.22	1.423077				2012	0.785	0.738919
2013	1.05	0.31	1.418919		Av d	Av r	2013	1.05	1.050002
2014	1.32				0.265	1.420998	2014	1.315	1.492051
2015	1.46						2015	1.58	2.120201
2016	1.51						2016	1.845	3.012802
2017							2017	2.11	4.281185
2018							2018	2.375	6.083555
2019							2019	2.64	8.644719
2020							2020	2.905	12.28413
2021							2021	3.17	17.45572
2022							2022	3.435	24.80454
2023							2023	3.7	35.2472
2024							2024	3.965	50.0862
2025							2025	4.23	71.17239



From the graph it looks like both models are equally accurate for the years 2011 to 2013 which justifies the reasoning for choosing these models. However as we extrapolate further it can be seen that model A is more accurate up to 2016



4 By looking at the charts it is seems that linear growth as predicted by model A seems more realistic. However if one looks at the actual numbers of shipments up to 2016 growth seems to be faster than arithmetic but slower than geometric. It seems that Arithmetic growth seems more realistic because technology is making rapid advancement and by 2025 smartphones may become obsolete. Also, the geometric model starts growing slowly but then grows very fast and

it would eventually exceed the world population. Presently the world population for 2018 is predicted to reach 7.6 billion. The model predicts shipments of about 6 billion.

However by 2020 the model predicts shipments of more than 12 billion which is very unrealistic.

1.3 Proof

TOK

Do all societies view investment and interest in the same way? What is your stance?

Answer: Students could research the reason as to why we charge interest on a loan and compare this with the perspectives in other societies such as where money in Islam is not regarded as an asset from which it is ethically permissible to earn a direct return. The Qur'an (2:279) sees interest as inequitable, as implied by the word "zulm" in Arabic which translates as oppression, exploitation, and the opposite of justice. There is no real loaning in Islam since lenders achieve ownership in the estates that they finance.

This allows students to view the perspectives of other societies and decide to what extent they agree with the charging of interest.

Investigation 10a

Note: There is no conceptual understanding for this investigation as it is meant to give an in context introduction to proof and reinforce the distinction between equations and identities.

Area ABCD	Area APTS	Area BPQT	Area STRD	Area TQCR
$(3+4)^2$	4^2	3 X 4	3X4	3^2
$(8+3)^2$	8^2	8X3	8X3	3^2
$(7+5)^2$	7^2	7X5	7X5	5^2

1 The area of square ABCD is the sum of the areas of square APTS, rectangles BPQT and STRD and square TQCR

2 $(3+4)^2 = 4^2 + 2(3\times 4) + 3^2$
$(8+3)^2 = 8^2 + 2(8\times 3) + 3^2$
$(7+5)^2 = 7^2 + 2(7\times 5) + 5^2$

3 $(a+b)^2 = a^2 + 2(a\times b) + b^2$

4 **Factual:** What do you call this relationship? Why?

Answer: This is called an identity because the relationship holds for any value of a and b.

Investigation 10b

Area ABCD	Area PQRS	Area $\triangle PBQ$	Area PQRS + 4 $\times$ Area $\triangle PBQ$
$(3+4)^2 = 49$	x^2	**6**	x^2+24

$(5+11)^2 = 256$	x^2	$\frac{55}{2}$	x^2+110
$(7+24)^2 = 961$	x^2	**84**	x^2+336

1 The area of square ABCD is the same as the sum of the areas of square PQRS and four triangles.

2 $x^2 = 49 - 24$

$x^2 = 256 - 110$

$x^2 = 961 - 84$

3 **Factual:** What do you call each of these relationships? Why?

Answer: These are called equations because in each case we can solve for x

4 5, 11, 25

5 $x^2 = (a+b)^2 - 4\left(\frac{ab}{2}\right)$

6 **Conceptual:** What do you call this relationship? Why?

Answer: This is called an identity because the relationship holds for any value of a and b.

7 **Conceptual:** How would you describe the difference between an equation and an identity?

Answer: An equation is true for particular values but an identity is true for all values.

TOK

What is the role of the mathematical community in determining the validity of a mathematical proof?

Answer: Knowledge claims in mathematics: Do proofs provide us with completely certain knowledge?

Can we talk about universal truth in mathematics?

Nature of mathematics and science: What is the difference between the Inductive method in Science and proof by induction in mathematics?

Investigation 11

The students are walked through this investigation by the student book.

Investigation 12

Note: This investigation allows students to distinguish the difference between a direct proof and proof by contradiction and so has no Conceptual understanding.

1 $5n+2 = 2k$ where $k \in \mathbb{Z}$

$\Rightarrow 5n = 2(k-1)$

$\Rightarrow 5n$ is even

But the product of two odd numbers is always odd as proved in Ex 1E question 2

Therefore since 5 is odd n must be even.

2 $5(2m+1)+2,\ m\in\mathbb{Z}$

3 $5(2m+1)+2=10m+7=2(5m)+7$

The sum of an even and odd number is always odd, so if n is an odd number $5n+2$ cannot be even.

4 In the second method we started by assuming that the statement was false and we ended up with a contradiction.

Investigation 13

Conceptual understanding:

Proof allows conclusions to be reached more elegantly than by alternative cumbersome ways.

1 $a+b=c \Rightarrow$ Triangle ABC becomes line segment ACB
$a+c=b \Rightarrow$ Triangle ABC becomes line segment ABC
$b+c=a \Rightarrow$ Triangle ABC becomes line segment BAC

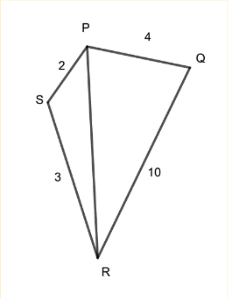
2 $2+4\geq AC \Rightarrow AC\leq 6$

3 $AC+3\geq 10 \Rightarrow AC\geq 7$

4 The two results contradict each other

5 $$\left.\begin{array}{l}2+3\geq BD \Rightarrow BD\leq 5\\ BD+4\geq 10 \Rightarrow BD\geq 6\end{array}\right\}\textit{contradiction}$$

6



$$\left.\begin{array}{l}2+3\geq PR \Rightarrow PR\leq 5\\ PR+4\geq 10 \Rightarrow PR\geq 6\end{array}\right\}\textit{contradiction}$$
$$\left.\begin{array}{l}2+4\geq SQ \Rightarrow SQ\leq 6\\ SQ+3\geq 10 \Rightarrow SQ\geq 7\end{array}\right\}\textit{contradiction}$$

7 **Factual:** What do you conclude from this investigation?

Answer: No matter what order we use for arranging the sides it is impossible to create a quadrilateral with sides of length 2,2,4 and 10 as shown by the contradictions in the answers above.

8 **Conceptual:** How else could you have come to the same conclusion?

Answer: Students could try to construct such a quadrilateral but that would be too cumbersome and this method of using the triangle inequality to prove by contradiction is more elegant.

This leads to the conceptual understanding: Proof allows conclusions to be reached more elegantly than by alternative cumbersome ways.

TOK

What do mathematicians mean by mathematical proof, and how does it differ from good reasons in other areas of knowledge?

Answer: In this section we have looked at proof, now we can compare mathematical proof to other areas of knowledge such as the natural sciences and the scientific method.

Is proof necessary in all areas of knowledge?

Which areas use proof? When is proof, or even reasoning, not necessary in other areas of knowledge?

International-mindedness

How did the Pythagoreans find out that $\sqrt{2}$ is irrational?

Answer: Root 2 was the hypotenuse of a right triangle of sides 1. Research what happened to Hippasus, and why.

Investigation 14

Note: This investigation is meant to give an in-context introduction to proof by induction and therefore has no conceptual understanding.

1 $1+2+1=4=2^2$; $1+2+3+2+1=9=3^2$; $1+2+3+4+3+2+1=16=4^2$

$1+2+3+4+5+4+3+2+1=5^2$

2 $1+2+3+4+5+6+5+4+3+2+1=6^2$

$1+2+3+4+5+6+7+6+5+4+3+2+1=7^2$

3 $1+2+3+\ldots+(n-1)+n+(n-1)+\ldots+3+2+1=n^2$

4 LHS

$$\begin{aligned}
&= 1+2+3+\ldots+(n-1)+n+(n-1)+\ldots+3+2+1 \\
&= 2(1+2+3+\ldots+(n-1))+n \\
&= 2\frac{(n-1)}{2}n+n \\
&= n(n-1)+n \\
&= n^2-n+n \\
&= n^2 \\
&= \text{RHS}
\end{aligned}$$

1.4 Counting principles and the binomial theorem

Investigation 15

Note: This investigation introduces factorial notation and so has no conceptual understanding.

1 **a** (2×1)

b $3\times(2\times1)=6$

c $5\times4\times3\times2\times1=120$

d $8\times7\times6\times5\times4\times3\times2\times1=40320$

2 The number of cups is represented by 1+2+3+…+6=21

3 The number of marbles in the last cup numbered 21 would be

$21\times20\times19\times\ldots\times3\times2\times1$

4 As the number of cups increases the number of marbles placed in each cup grows big very quickly.

5 $n\times(n-1)\times(n-2)\times\ldots\times3\times2\times1$

Investigation 16a

1 6 arrangements

2 Students would probably list all the ways at this stage

3 24 ways

4 6 + 24 =30

5 She would have to send an invitation that is a duplicate of one of the 24 invitations sent to friends because there are no other arrangements of the four objects. i.e. there is no other permutation.

TOK

How many different tickets are possible in a lottery?

What does this tell us about the ethics of selling lottery tickets to those who do not understand the implications of these large numbers?

Answer: You might want to use stimuli to start a discussion that might result in a class debate or a blog post such as:

Are lotteries marketed to people who don't have an emergency fund, are low on finances and are bad at math?

Is it ethical to sell hope if the odds are 1 in 45 million?

You are probably not going to win, but winning the lottery is not the point, it is the thrill?

Does the thrill generate and addiction to gambling?

Is gambling a tax on the less intelligent?

Investigation 16b

1 She now can choose the first photo in five ways.

2 The second photo can be chosen in four ways.

3 She can choose 2 photos out of 5 in 10 ways.

4 Students may not realize that there is a difference, but they can be guided to think about the condition that there is no preference as to which goes on which invitation. The number of ways of Choosing 2 photos out of 5 is called a **combination** of 2 out of 5.

Investigation 16c

1 Although there are 10 permutations there are only five combinations as shown by the five different colours

2 One would obtain 10 permutations but again only five combinations.

3 If all the permutations are considered then for every combination of three letters six permutations are possible.

4 There are 5 ways of choosing the first letter, 4 ways of choosing the second letter and 3 ways of choosing the third letter giving $5 \times 4 \times 3$ which can be written as $\frac{5!}{3!} = \frac{5!}{(5-3)!}$. But since for each distinct group of 3 letters there are 3! ways of arranging the letters we need to divide by 3!

So the number of ways of choosing 3 letters out of 5 will be $\frac{5!}{3!(5-3)!} = 10$

Investigation 17

$(1+x)^i$	Constant	Coefficient of x	Coefficient of x^2	Coefficient of x^3	Coefficient of x^4	Coefficient of x^5
$(1+x)^0$	1	-	-	-	-	-
$(1+x)^1$	1	1	-	-	-	-
$(1+x)^2$	1	2	1	-	-	-
$(1+x)^3$	1	3	3	1	-	-
$(1+x)^4$	1	4	6	4	1	-
$(1+x)^5$	1	5	10	10	5	1

1 The constant term is always 1

The coefficients of x in the second column are the positive integers

The coefficients of x^2 in the third column are the triangle numbers encountered in investigation 4.

2

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

3 There is a line of symmetry going down the middle of the numbers

Each row starts with and the next number is the sum of the two numbers above it to either side.

4 1 6 15 20 15 6 1

5
$$\begin{aligned}(1+x)^6 &= (1+5x+10x^2+10x^3+5x^4+x^5)(1+x)\\ &= 1+6x+15x^2+20x^3+15x^4+6x^5+x^6\end{aligned}$$

Investigation 18

Conceptual understanding:

The binomial theorem uses combinations to calculate the coefficients in the expansion and these coefficients display symmetry about the centre.

1 The constant term is obtained by multiplying all the 1's and not choosing any x's

2 By choosing one of the x's from the three factors $(1+x)$ and multiply it by the 1's in the other factors. There are three ways of choosing the x and each time you multiply it by the 1's from the other factors.

3 This time you choose two x's from the three factors and multiply by the 1 in the other factor. There are three ways of choosing two x's out of three.

4 There is only one way of obtaining this coefficient and that is by choosing all the x's and multiplying them.

5 $(1+x)^3 = (1+x)(1+x)(1+x) = {}^3C_0 \times 1 + {}^3C_1 \times x + {}^3C_2 \times x^2 + {}^3C_3 \times x^3$

6 $(a+x)^3 = (a+x)(a+x)(a+x) = {}^3C_0 \times a^3 + {}^3C_1 \times a^2 \times x + {}^3C_2 \times a \times x^2 + {}^3C_3 \times x^3$

7
$$\begin{aligned}(a+x)^n &= \underbrace{(a+x)(a+x)(a+x)\ldots(a+x)}_{n \text{ factors}}\\ &= {}^nC_0 \times a^n + {}^nC_1 \times a^{n-1} \times x + {}^nC_2 \times a^{n-2} \times x^2 + \ldots + {}^nC_r \times a^{n-r} \times x^r + \ \ldots \ + {}^nC_{n-1} \times a \times x^{n-1} + {}^nC_n \times x^n\end{aligned}$$

8 Conceptual: How does the binomial theorem use combinations to obtain a binomial expansion?

Answer (this is the conceptual understanding): The binomial theorem uses combinations to calculate the coefficients in the expansion and these coefficients display symmetry about the centre.

9 Conceptual: How is binomial theorem related to Pascal's triangle?

Answer: The coefficients in the binomial theorem for power n are the numbers in nth row of pascal's triangle (assuming top row is row 0).

Investigation 19

Conceptual understanding:

Numbers and formulae can appear in different but equivalent forms or representations to establish identities.

1 $(1+x)^n = {}^nC_0x^0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$

2 **a** 5, 7, 11

b the number of terms for a power 2n is 2n+1

c 4, 6, 8

d The number of terms for a power 2n-1 is

3 **a, c**

$$(1+x)^5 = {}^5C_0x^0 + {}^5C_1x + {}^5C_2x^2 + {}^5C_3x^3 + {}^5C_4x^4 + {}^5C_5x^5$$
$$(1+x)^6 = {}^6C_0x^0 + {}^6C_1x + {}^6C_2x^2 + {}^6C_3x^3 + {}^6C_4x^4 + {}^6C_5x^5 + {}^6C_6x^6$$

To show symmetry

$${}^5C_1 = \frac{5!}{5!(5-1)!} = 5, \quad {}^5C_4 = \frac{5!}{4!(5-4)!} = 5$$

$${}^5C_2 = \frac{5!}{2!(5-2)!} = 10, \quad {}^5C_3 = \frac{5!}{3!(5-3)!} = 10$$

$${}^6C_1 = \frac{6!}{1!(6-1)!} = 6, \quad {}^6C_5 = \frac{6!}{5!(6-5)!} = 6$$

$${}^6C_2 = \frac{6!}{2!(6-2)!} = 15, \quad {}^6C_4 = \frac{6!}{4!(6-4)!} = 15$$

To show each line starts and ends with 1

$${}^5C_0 = \frac{5!}{0!(5-0)!} = 1, \quad {}^5C_5 = \frac{5!}{5!(5-5)!} = 1 \quad {}^6C_0 = \frac{6!}{0!(6-0)!} = 1, \quad {}^6C_6 = \frac{6!}{6!(6-6)!} = 1$$

To show: ${}^6C_1x = {}^5C_0x^0 + {}^5C_1x$

LHS: =6

$$\text{RHS} = \frac{5!}{0!5!} + \frac{5!}{1!4!} = \frac{5! + 5\times 5!}{5!} = \frac{5!(1+5)}{5!} = 6$$

Similarly ${}^6C_2x^2 = {}^5C_1x + {}^5C_2x^2$

LHS = 15

$$\text{RHS} = \frac{5!}{1!4!} + \frac{5!}{2!3!} = \frac{2\times 5! + 4\times 5!}{2!4!} = \frac{5!(2+4)}{2!4!} = \frac{5\times 6}{2} = 15$$

Similarly ${}^6C_3x^3 = {}^5C_2x^2 + {}^5C_3x^3$

LHS = 20

$$\text{RHS} = \frac{5!}{2!3!} + \frac{5!}{3!2!} = 2\times\frac{5!}{3!2!} = 2\times\frac{5\times 4}{2!} = 20 \text{ etc...}$$

b, c

$$(1+x)^n = {}^nC_0x^0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_{r-1}x^{r-1} + {}^nC_rx^r + \ldots + {}^nC_nx^n$$
$$(1+x)^{n+1} = {}^{n+1}C_0x^0 + {}^{n+1}C_1x + {}^{n+1}C_2x^2 + \ldots\ldots + {}^{n+1}C_rx^r + \ldots + {}^{n+1}C_nx^n$$

$$^{n}C_{0} = \frac{n!}{0!(n-0)!} = 1 = \frac{(n+1)!}{0!(n+1-0)!} = {}^{n+1}C_{0}$$

$$^{n}C_{n}x^{n} = \frac{n!}{n!(n-n)!} = 1 = \frac{(n+1)!}{(n+1)!(n+1-n-1)!} = {}^{n+1}C_{n+1}$$

To prove symmetry we need to show that ${}^{n}C_{r} = {}^{n}C_{n-r} \Rightarrow {}^{n}C_{r} - {}^{n}C_{n-r} = 0$

$$^{n}C_{r} - {}^{n}C_{n-r} = \frac{n!}{r!(n-r)!} - \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!} - \frac{n!}{(n-r)!r!} = 0$$

We are now required to show that ${}^{n+1}C_{r}x^{r}\ {}^{n}= C_{r-1}x^{r-1} + {}^{n}C_{r}x^{r}$

$$^{n}C_{r-1}x^{r-1} + {}^{n}C_{r}x^{r} = \frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r+1)\times(n-r)!} + \frac{n!}{r\times(r-1)!(n-r)!}$$

$$= \frac{n!\times r + n!\times(n-r+1)}{r!(n-r+1)!}$$

$$= \frac{n!(r+n-r+1)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!} = {}^{n+1}C_{r}x^{r}$$

d **Conceptual:** How can you explain the patterns in Pascal's triangle by considering the general expansion of the binomial expansion?

Answer: The above answers using the expansions for $(1+x)^{n}$ and $(1+x)^{n+1}$ and algebra have allowed us to prove the three properties seen in Pascal's triangle.

(This leads to the conceptual understanding): Numbers and formulae can appear in different but equivalent forms or representations to establish identities

TOK

Why do we call this Pascal's triangle when it was in use before Pascal was born?

Are mathematical theories merely the collective opinions of different mathematicians, or do such theories give us genuine knowledge of the real world?

Answer: Blaise Pascal is credited with Pascal's Triangle after he wrote about in a treatise called "The Arithmetic triangle", but the properties of "Pascal's Triangle" have been known in a number of different cultures long before Pascal. (e.g the Chinese mathematician

Yang Hui, the Indian mathematician Pangala and Persian poet and mathematician Omar Khayyam).

TOK

Is it possible to know things about which we can have no experience, such as infinity?

Answer: Consider the debate over the validity of the notion of "infinity".

Investigation 20

1 $(1-x)^{n} = 1 + nx + \frac{n(n+1)}{2!}x^{2} + \frac{n(n+1)(n+2)}{3!}x^{3} + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^{r} + \ldots$

Required to show that $(1-x)^{-n} = \sum_{0}^{\infty} {}^{n+r-1}C_r x^r$

When $r = 0$, RHS $= {}^{n-1}C_0 x^0 = \dfrac{(n-1)!}{0!(n-1-0)!} = 1$

When $r > 0$, RHS $= \dfrac{(n+r-1)(n+r-2)....(n)(n-1)!}{r!(n+r-1-r)!}x^r$

$= \dfrac{(n+r-1)(n+r-2)....(n+1)(n)(n-1)!}{r!(n-1)!}x^r$

$= \dfrac{(n+r-1)(n+r-2)....(n+1)(n)}{r!}x^r$

$= \dfrac{n(n+1)(n+2)...(n+r-1)}{r!}x^r$

Which corresponds to the general term in the expansion given.

Now we can enter values for r to obtain:

$$(1-x)^{-n} = 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$$

2

$$(1-x)^{-1} = 1 + \frac{1}{1!}x + \frac{1(1+1)}{2!}x^2 + \frac{1(1+1)(1+2)}{3!}x^3 + \frac{1(1+1)(1+2)(1+3)}{4!}x^4 + \frac{1(1+1)(1+2)(1+3)(1+4)}{5!}x^5 + \ldots$$
$$= 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$$

$$(1-x)^{-2} = 1 + \frac{2}{1!}x + \frac{2(2+1)}{2!}x^2 + \frac{2(2+1)(2+2)}{3!}x^3 + \frac{2(2+1)(2+2)(2+3)}{4!}x^4 + \frac{2(2+1)(2+2)(2+3)(2+4)}{5!}x^5 + \ldots$$
$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \ldots$$

$$(1-x)^{-3} = 1 + \frac{3}{1!}x + \frac{3(3+1)}{2!}x^2 + \frac{3(3+1)(3+2)}{3!}x^3 + \frac{3(3+1)(3+2)(3+3)}{4!}x^4 + \frac{3(3+1)(3+2)(3+3)(3+4)}{5!}x^5 + \ldots$$
$$= 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \ldots$$

$$(1-x)^{-4} = 1 + \frac{4}{1!}x + \frac{4(4+1)}{2!}x^2 + \frac{4(4+1)(4+2)}{3!}x^3 + \frac{4(4+1)(4+2)(4+3)}{4!}x^4 + \frac{4(4+1)(4+2)(4+3)(4+4)}{5!}x^5 + \ldots$$
$$= 1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 + \ldots$$

$$(1+x)^{-1} = (1-(-x))^{-1}$$

Using the result for the first expansion we obtain:

$$(1+x)^{-1} = 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + (-x)^5 + \ldots$$
$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$$

3

$(1-x)^i$	Constant	Coefficient of x	Coefficient of x^2	Coefficient of x^3	Coefficient of x^4	Coefficient of x^5
$(1-x)^{-1}$	1	1	1	1	1	1 ...
$(1-x)^{-2}$	1	1	3	4	5	6...
$(1-x)^{-3}$	1	3	6	10	15	21

$(1-x)^{-4}$	1	4	10	20	35	56
$(1+x)^{-1}$	1	-1	1	-1	1	-1

4

$$(1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\ldots+\frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-r+1)}{r!}x^r+\ldots$$

Required to show that $(1+x)^{\alpha}=\sum_{r=0}^{\infty}{}^{\alpha}C_r x^r$

When $r=0$, RHS $={}^{\alpha}C_0x^0=\dfrac{\alpha!}{0!(\alpha-0)!}=1$

When $r>0$, RHS $=\dfrac{\alpha!}{r!(\alpha-r)!}x^r=\dfrac{\alpha(\alpha-1)\ldots(\alpha-r+1)}{r!}$

Which corresponds to the general term of the expansion given.

5

$$\sqrt{(1+x)}=(1+x)^{\frac{1}{2}}=1+\frac{\left(\frac{1}{2}\right)}{1!}x+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}x^4$$
$$+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)}{5!}x^5+\ldots$$
$$=1+\left(\frac{1}{2}\right)\frac{x}{1!}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^2}{2!}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{3!}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{x^4}{4!}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\frac{x^5}{5!}$$
$$=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\frac{7}{256}x^5+\ldots$$

$$\sqrt{(1-x)}=(1+(-x))^{\frac{1}{2}}=1+\frac{\left(\frac{1}{2}\right)}{1!}(-x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-x)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}(-x)^4$$
$$+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)}{5!}(-x)^5+\ldots$$
$$=1-\left(\frac{1}{2}\right)\frac{x}{1!}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^2}{2!}-\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{3!}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{x^4}{4!}-\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\frac{x^5}{5!}+\ldots$$
$$=1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{5}{128}x^4-\frac{7}{256}x^5+\ldots$$

TOK

"Mathematics may be defined as the economy of counting. There is no problem in the whole of mathematics which cannot be solved by direct counting." (E. Mach)

To what extent do you agree with this quote?

Answer: You might want to view the extraordinary links between Pascal's Triangle and the coefficients of polynomials and if this is this just a coincidence.

Is the nature of mathematics more profound than we realise?

Developing inquiry skills

Return to the chapter opening problem. The enclosed area of the Koch snowflake can be found using the sum of an infinite series.

In the second iteration, since the sides of the new triangles are $\frac{1}{3}$ the length of the sides of the original triangle, their areas must be $\left(\frac{1}{3}\right)^2 = \left(\frac{1}{9}\right)$ of its area.

If the area of the original triangle is 1 square unit, then the total area of the three new triangles is $3\left(\frac{1}{9}\right)$.

i Find the total area for the third and fourth iterations.

Answer: The total area for the third iteration is $12\left(\frac{1}{9}\right)^2$.

The total area for the fourth iteration is $48\left(\frac{1}{9}\right)^2$.

ii How can you use what you have learned in this section to find the total area of the Koch snowflake?

Answer: This makes the series: $1 + 3\left(\frac{1}{9}\right) + 12\left(\frac{1}{9}\right)^2 + 48\left(\frac{1}{9}\right)^2 + \ldots$

Since this is a converging geometric series with $r = \frac{1}{9}$,

$$S_\infty = 1 + \frac{u_1}{1-r}$$

$$S_\infty = 1 + \frac{\frac{1}{3}}{1-\frac{4}{9}}$$

$$S_\infty = 1 + \frac{\frac{1}{3}}{\frac{5}{9}}$$

$$S_\infty = 1 + \frac{3}{5}$$

$$S_\infty = \frac{8}{5}$$

iii How does the area of a Koch snowflake relate to the area of the initial triangle?

Answer: So, no matter the size of the initial triangle, the total area of the Koch Snowflake is $\frac{8}{5}$ its area.

Modelling and investigation activity: The Tower of Hanoi

Approaches to Learning: Thinking Skills, Communicating, Research

Exploration Criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB Topic: Sequences

Introduction

The Towers of Hanoi problem is a challenging ancient puzzle that prompts students to engage in problem solving. Students should understand that struggling with a problem, and possibly having to rethink their approach, is the nature of mathematics and a normal part of learning. Persevering with a challenging problem will grow new connections in the brain and, over time, makes difficult tasks easier.

In terms of preparing for an exploration - for the teacher and student this should start from chapter 1. This chapter on sequences and series, proof and binomial expansions is a great base for explorations. The earlier the IA and the IA criteria are introduced the better, as this will encourage students to start to think of ideas and to make connections. The Tower of Hanoi problem is a 'classic' mathematics problem. This could be an issue, as students could simply use what is already available online and in books. With this in mind the problem is approached here by encouraging personal engagement with the problem (Criterion C) rather than seek a solution online. Students should consider the possible approaches to answer a challenging mathematical problem like this. They are also required to consider how they communicate mathematically (Criterion B) and then are asked to think about possible extensions and to research other avenues of exploration (Criterion E: Use of mathematics).

The problem

The Tower of Hanoi problem, also called Towers of Hanoi or Towers of Brahma, involves three vertical pegs and a set of different sized disks with holes through their centres. The Tower of Hanoi problem is widely believed to have been invented in 1883 by the French mathematician Édouard Lucas (though his role in its invention has been disputed). Ever popular, made of wood or plastic, the Tower of Hanoi can be found in toy shops around the world.

Lucas apparently spread the legend that helped popularize the game by including a written account in each of the toy boxes sold of the Brahmin monks moving 64 golden disks between three poles for many centuries with the legend saying that when they completed the puzzle the world would end! The legend varies over time and place, being set either in a temple or a monastery in Vietnam or India. In some versions of the legend the monks are only allowed to make one move per day.

The history of the problem is interesting as it gives context and often a rationale for studying the problem.

To encourage students to engage with the problem, you could ask:

What is the history and legend behind the problem?

What is the significance of the 64 disks to the legend?

Why research the history of the problem?

Explore the problem

The first thing for students to do is to have a 'play' with the problem. From this, since 64 disks makes the problem too time consuming, one possible approach is to start with smaller numbers of disks and build up a sequence so as to develop a formula. This can lead to either finding a recursive formula, an explicit formula or a graphical solution.

Suggestions of simulations that could be used are:

[Mathsisfun (https://www.mathsisfun.com/games/towerofhanoi.html)](https://www.mathsisfun.com/games/towerofhanoi.html)

[Webgamesonline (http://www.webgamesonline.com/towers-of-hanoi/)](http://www.webgamesonline.com/towers-of-hanoi/)

[haubergs (http://haubergs.com/hanoi)](http://haubergs.com/hanoi)

If online simulations are not available, then a physical representation of the problem could be used.

When $n = 3$ the minimum number of moves required is 7.

When $n = 4$ the minimum number of moves required is 15.

To encourage discussion, you could ask:

How do you know these are minimum values?

This could be established through multiple students finding the same answer.

As part of the exploration, you could ask students to think about what representations (diagram, table or other form of representation) they could use to display the individual moves needed to solve the problem. They could then try to represent the moves made using their chosen method.

Possible methods of representation involve diagrams of the different moves, a table that represents the moves of individual disks, a graph theory approach, etc.

Try and test a rule

If the solution is arithmetic, then you could use the result for n = 4 and the common difference to find the result for n = 5.

The common difference between the number of moves for n = 3 and n = 4 is 15 - 7 = 8, so the common difference between n = 4 and n = 5 would also be 8.

The number of moves for n = 5 would be 15 + 8 = 23.

Using a simulator, the minimum number of moves when n = 5 is 31, not 23.

The solution does not follow an arithmetic sequence.

Find more results

For n = 1 the minimum number of moves is 1.

For n = 2 the minimum number of moves is 3.

To put the results into context, you could encourage students to talk about the method being used to solve the problem.

The method is to move the disks so that all but the largest disk have been assembled in their correct order on peg B. The largest disk is then moved to peg C from peg A and then the remaining disks are assembled on top of this piece using the same number of moves as before.

n (the number of disks)	M_n (the minimum number of moves needed for n disks)
1	1
2	3
3	7
4	15
5	31

Some students may be able to spot the formula ($M_n = 2^n - 1$) from this data. Others may require more work.

As an **extension**, you could ask students to use a graphing package to graph the data from their table, with the number of disks, n, on the horizontal axis, and the minimum number of moves, M, on the vertical axis.

You could ask:

How could you use the graph to find a formula?

Students may be able to spot the type of expression that could be used to fit through the points on the graph.

Try a formula

To give students further guidance, you could ask:

What must happen before the largest disk can be moved to peg C?

Before the largest disk can be moved to peg C, the other disks need to be assembled in order on peg B.

It would take a minimum of 7 moves to get the 3 pieces on peg B as shown.

It would then take 1 move to move the largest disk from peg A to peg C.

As the pieces need to reassemble as they are it would take another 7 moves to move the 3 smaller disks to peg C.

Therefore, the total number of moves is 7 + 1 + 7 = 15.

This method will set up the recursive formula of the solutions:

$$M_{n-1} = 2 \times M_n + 1$$

Where M_n is the minimum number of moves needed for n disks.

Make sure that students carefully consider the notation in the formula and that the variables in their formula are well defined.

This is a recursive formula. It uses the minimum number of moves needed to solve an n disk puzzle to find the minimum number of moves needed for an (n + 1) disk puzzle.

To check that the formula works, students could try to solve n = 6 and check the result against the formula.

The problem with a recursive formula is that you need to have solved all previous iterations of the problem in order to solve the next one.

Try another formula

The relationship is not geometric because there is not a common ratio between terms.

Some students may spot the relationship to 2^n. Others may not. Adding 1 to each term might help students to recognize the relationship.

2, 4, 8, 16, 32

The formula could be written as $M_n = 2^n - 1$

An explicit formula differs from a recursive formula because it does not require you to know the previous terms. It is possible just to substitute in the value of n.

For n = 64, the minimum number of moves needed is:

$M_{64} = 2^{64} - 1$ = 18,446,744,073,709,600,000.

Even if the monks take one second per move this would take more than 584.9 billion years, which is longer than the history of our known universe (approximately 13.8 billion years).

Extension

The suggested extension activities look at different versions of the Towers of Hanoi problem, as well as exploring recursive formulae.

There are many classic puzzles like this that involve sequences and series and these can lead to starting points for explorations. However, with this and other classic problems it is important that students do not simply regurgitate what is already available, but that they instead engage with the problem. You could also consider extensions and additional research on top of the regular problem.

You could ask students what other ‘classic problems’ in mathematics they know and ask them to explore these problems.

2 Representing relationships: functions

Essential understanding

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables visually and symbolically as graphs, equations and/or tables represents different ways to communicate mathematical ideas.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function of equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this "window" can show more or less to best suit our needs.
- Equivalent representations of quadratic functions can reveal different characteristics of the same relationship.
- Functions represent mappings that assign to each value of the independent variable (input) one and only one dependent variable (output).

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
A function assigns to each input value one and only one output value.	Investigation 1
Different representations of functions, symbolically and visually as graphs, equations and tables, provide different ways to communicate mathematical relationships.	Investigation 1
The vertical line test (on a graph) helps to identify relations that represent functions because if a vertical line intersects the graph in more than one point, then at least one x-value has more than one y-value, hence it does not represent the graph of a function.	Investigation 2

Conceptual understandings	Investigation
Equivalent representations of quadratic functions can reveal different characteristics of the same relationship.	Investigation 3
The parameters of a function correspond to geometrical features of a graph, such as its width and its concavity.	Investigation 4
Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.	Investigation 5
The spatial frame of reference affects the visible part of a function and changing this "window" using technology can show more or less of the function to best suit our needs.	Investigation 5
If the horizontal lines drawn through the graph of an onto function intersect the graph of the function in at least one point, this indicates that the domain is mapped to is the set of images of the domain, i.e. the range.	Investigation 12
If $f(x)=f(-x)$ then the function is even; if $f(-x)=-f(x)$ then the function is odd.	Investigation 13
The domain of a function that consists of more than one function would be the set common to all functions; for the quotient of functions, any values that make the denominator zero would be excluded from the intersection of the sets.	Investigation 15
Inverse functions have reflection symmetry about the line y=x, since the reflection line $y=x$ bisects the line segment joining any two corresponding points between the graph of the function and its inverse.	Investigation 18
Only functions both one-to-one and onto have inverse functions, since a many-to-one function has a one-to-many inverse.	Investigation 19
The graph of the *absolute value of a function* remains unchanged for non-negative *y*-values of the function and reflected in the *x*-axis for negative *y*-values since the range of the function is the set of positive real numbers. The graph of the *function of the absolute value* remains unchanged for non-negative *x*-values of the function, and for negative *x*-values; the graph for its non-negative *x*-values is reflected in the *y*-axis.	Investigation 21
The order in which transformations on a graph of a function are carried out influence the final result.	Investigation 25

Syllabus sections covered in this chapter:

- SL2.1*
- SL2.2*
- SL2.3*
- SL2.4*
- SL2.5

- SL2.6
- SL2.7
- SL2.8
- SL2.10
- SL2.11
- AHL1.11
- AHL2.12
- AHL2.13
- AHL2.14
- AHL2.15
- AHL2.16

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and the development of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 73: Sketching graphs, finding intersection points, changing the form of a quadratic equation.	Page 88: Example 8 Page 100: Example 21 Page 110: Example 26 Page 117: Example 30 Page 129: Example 37	Page 87: Example 7 Page 88: Example 8 Page 91: Example 11 Page 96: Example 16 Page 97: Example 17 Page 99: Example 20 Page 119: Example 31	Pages 80, 101, 108, 116, 139

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 139	Page 142	N/A

2.1 Functional relationships

Investigation 1

Conceptual understandings:

A function assigns to each input value one and only one output value.

Different representations of functions, symbolically and visually as graphs, equations and tables, provide different ways to communicate mathematical relationships.

1 **Factual:** What is a relation?

Answer: An ordered pair, or set of ordered pairs, defines a relation.

2 The relations in the second column have no repeating x-values, whereas the relations in the first column do.

3 **Conceptual:** What is a function?

Answer (this is a conceptual understanding): A function assigns to each input value one and only one output value

4 Students should produce their own examples of functions using either a mapping diagram, graphs, sets, or algebraic functions.

5 Functions and relations are used to describe mathematical relationships between variables.

6 Students should think about situations in which it would detrimental to have one input turn out two or more different outputs, e.g., interest problems.

7 **Conceptual:** Why is it useful to be able to express functions in different forms?

Answer (this is a conceptual understanding): Different representations of functions, symbolically and visually as graphs, equations and tables, provide different ways to communicate mathematical relationships.

Investigation 2

Conceptual understanding:

The vertical line test (on a graph) helps to identify relations that represent functions because if a vertical line intersects the graph in more than one point, then at least one x-value has more than one y-value, hence it does not represent the graph of a function.

1 **i** 1 and 4

ii 2 and 3

2 **i** The vertical lines intersect the graphs only once.

ii The vertical lines intersect the graphs more than once.

3 If vertical lines drawn on the graph of a relation intersect the graph only once, then the graph of the relation is the graph of a function.

4 The vertical lines have equations in the form of $x = a, a \in \mathbb{R}$. This means that all points on the line have the same x-value, that is, for one x-value there is more than one y-value.

5 **Conceptual:** Why is using vertical lines to test if the graph of a relation represents a function effective?

Answer (this is the conceptual understanding): The vertical line test (on a graph) helps to identify relations that represent functions because if a vertical line intersects the graph in more than one point, then at least one *x*-value has more than one *y*-value, hence it does not represent the graph of a function.

TOK

Around the world you will often encounter different words for the same object, like trapezium and trapezoid or root and surd. Sometimes more than one type of symbol might have the same meaning such as interval and set notation.

To what extent does the language we use shape the way we think?

Answer: Is it OK to only use one type of mathematical and/or national term?
Whose job is it to impart knowledge and understanding?
How has technology influenced the notation that we use?
You might want to research how many words that Eskimos have for snow.
How do you think that this use of language affects understanding of space, time, colours, and objects?

Developing inquiry skills

Refer back to the opening problem. How does this section help you to answer the first two questions of the opening problem?

How many apples are in each box?

Answer: You could determine a function which related the volume of the box to the approximate number of apples in the box. To do this, you would need to know the approximate volume one apple occupies.

How can the value of one apple be determined?

Answer: The value of one apple could be determined by dividing the price of the box by the number of apples in the box (as given by your function found in the previous question).

2.2 Special functions and their graphs

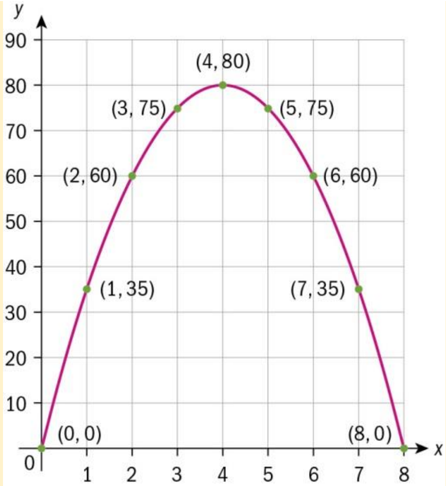
Investigation 3

Conceptual understanding:

Equivalent representations of quadratic functions can reveal different characteristics of the same relationship.

1 The 2nd differences are constant, therefore this table represents a quadratic function. (This links to chapter 1.)

2



3 $y = -5x(x-8)$

4 8 s

5 80 m; 4 s

6 $y = -5(x-4)^2 + 80$

7 **Factual:** Which forms of a quadratic function highlight different information in a given problem?

Answer: The vertex form shows its maximum height and the time it takes to reach this height. The intercept form shows the times when the projectile was thrown and when it landed, so you can tell how long it was in the air.

8 **Conceptual:** Why is it useful to be able to express quadratic functions in different equivalent forms?

Answer (this is the conceptual understanding): Equivalent representations of quadratic functions can reveal different characteristics of the same relationship.

Investigation 4

Conceptual understanding:

The parameters of a function correspond to geometrical features of a graph, such as its width and its concavity.

1 Answers will vary.

2 The parameter *h* shifts the graph of $y = x^2$ horizontally and *k* shifts the graph vertically.

3 Answers will vary.

4 **i** $a > 0$

ii $a < 0$

5 The function is linear.

6 k will be either the maximum or minimum value of the function, depending on whether $a < 0$ or $a > 0$ respectively.

7 **Conceptual:** What features of a graph of a function do the parameters represent?

Answer (this is the conceptual understanding): The parameters of a function correspond to geometrical features of a graph, such as its width and its concavity.

International-mindedness

Cartesian coordinates are named after Frenchman Rene Descartes.

Investigation 5

Conceptual understandings:

Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.

The spatial frame of reference affects the visible part of a function and changing this "window" using technology can show more or less of the function to best suit our needs.

1 The functions and domains used to create the figure in Investigation 5 are given here.

Blue hairline: $y = -(x-4)^2 + 9; 2 \le x \le 6$

Red chin: $y = (x-4)^2 + 3; 2 \le x \le 6$

Pink shoulders: $y = -\frac{3}{4}(x-4)^2 + 3; 2 \le x \le 6$

Violet top of head: $y = -(x-4)^2 + 11; 2 \le x \le 6$

Smile: $y = 2(x-4)^2 + 4.5; 3.5 \le x \le 4.5$

Left eye: $y = -(x-3)^2 + 7; 2.5 \le x \le 3.5$

Right eye: $y = -(x-5)^2 + 7; 4.5 \le x \le 5.5$

Left earring yellow part: $y = -(x-2)^2 + 5; 1.5 \le x \le 2.5$

Left earring grey part: $y = -(x-2)^2 + 4.5; 5.5 \le x \le 6$

Right earring yellow part: $y = -(x-6)^2 + 5; 5.5 \le x \le 6.5$

Right earring grey part: $y = -(x-6)^2 + 4.5; 5.5 \le x \le 6.5$

2 **Conceptual:** Why is it helpful to move between different forms of the quadratic to model the given problem?

Answer (this is a conceptual understanding): Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.

3 $x : [-1, 10]; y : [-1, 12]$

4 **Conceptual:** When using technology, how does the choice of viewing window affect your ability to communicate patterns effectively?

Answer (this is the conceptual understanding): The spatial frame of reference affects the visible part of a function and changing this "window" using technology can show more or less of the function to best suit our needs

5 Students should check if their functions produce the same graphs.

International-mindedness

The development of functions bridged many countries including Rene Descartes (France), Gottfried Wilhelm Leibnitz (Germany) and Leonhard Euler (Switzerland).

The notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries, you can ask students "how did the notation we use today become internationally accepted?"

Reflect How do you think you can check your answer to the horizontal asymptote by using values of x?

Answer: There should be no values of x for which the function attains the value of the horizontal asymptote; i.e. if the asymptote is $y = a$, there should be no x in the domain for which $f(x) = a$.

Investigation 6

This investigation will allow students to see the need to restrict the domain of certain functions.

1

x	$f(x)$
2	1
3	3
4	$1+2\sqrt{2}$
5	$1+3\sqrt{2}$
6	5

2 The answers are negative under the radical sign.

3 Any values that make the radical expression negative, i.e., any values less than 2.

4 Domain=$\{x|x\geq 2\}$; Range=$\{y|y\geq 0\}$

Investigation 7

Students will realize the difference between the relation $y^2 = x$ and the functions $y = \sqrt{x}$ and $y = -\sqrt{x}$

1 $y^2 = x$

2 No, as the same x-value gives two different y-values.

3 Yes, as for every x-value there is only one y-value.

Investigation 8

Students will realize that the graph of a function can have features of two or more different functions.

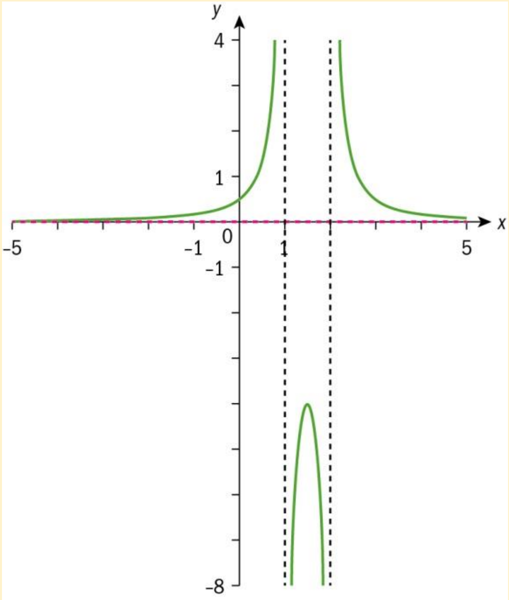
a $x=1$; $x=2$

b $\{x|x\neq 1, x\neq 2\}$

c 0

d $x=1$; $x=2$; $y=0$ as seen in the sketch below.

e



The graph contains features of both rational and quadratic functions.

International-mindedness

The word *asymptote* is derived from the Greek *asymptotos,* which means "not falling together".

TOK

How is mathematics related to reality?

Answer: You might want to consider whether mathematics is invented or discovered, whether it is sure or factual or both plus, whether mathematics is independent of culture.

If we accept that mathematics is discovered as opposed to invented, we have to deal with the fact that pure mathematics does not rely on any prior real-life situation. How can you explain this? You could contrast this with the application of mathematics to the real world.

Investigation 9

Students will investigate the effects of the parameters *h* and *k* on the graph of an absolute value function of the form $y=|x-h|+k$ for $h,k\in\mathbb{R}$

1 Answers will vary.

2 The parameter h shifts the graph horizontally and k shifts the graph vertically.

3 The graph is wider or narrower, depending on the value of a.

4 The minimum or maximum is (h, k). When $a<0$ it is a maximum, and when $a>0$ it is a minimum.

5 Domain: Reals; Range: For $a<0$, the range is $\{y \mid y\le k\}$. For $a>0$, the range is $\{y \mid y\ge k\}$.

TOK

"The object of mathematical rigour is to sanction and legitimize the conquests of intuition" - Jacques Hadamard

Do you think that studying the graph of a function contains the same level of mathematical rigour as studying the function algebraically?

Answer: A good opportunity to debate, in teams or pairs. first, define mathematical rigour. Opening questions such as what can analysis offer us that graphing a function cannot? What are the strengths of a function?

Reflect: How would you solve Example 20 algebraically?

Answer: $\left|\frac{1}{2x-1}\right| < 1 \Rightarrow -1 < \frac{1}{2x-1} < 1$

$Case\ 1: \ 0 < \frac{1}{2x-1} < 1 \Rightarrow 1 < 2x-1 \Rightarrow x > 1$

$Case\ 2: \ -1 < \frac{1}{2x-1} < 0 \Rightarrow 1-2x > 1 \ (inequality\ reverses\ because\ 2x-1 < 0)$

$\Rightarrow x < 0$

TOK

Is zero the same as nothing?

Answer: A good debate for pairs or a small group to then present their opinions to the rest of the class. You might like to reference "Coke zero" or "Ground Zero" juxtaposed with zero money in the bank. You might like to conclude with 'zero' is a number while 'nothing' is a concept.

Investigation 10

The purpose of this investigation is to illustrate how the graphs of functions can model every day objects.

1

$f1 = k(x)$

$f2 = v(x)$

$f3 = g(x)$

$f4 = p(x)$

$f5 = j(x)$

$f6 = t(x)$

$f7 = h(x)$

2 Answers will vary.

Developing inquiry skills

Let's return to the opening problem. How do the ideas in this section help you?

Answer: There are many possible ways that ideas in section 2.2 could relate to the opening problem. Here are a few that might start students off:

- Of the functions that you have studied, which type might best model the relationship between price of a box of apples and number of apples in the box?
- What function might best model the relation between volume of the box and number of apples, or the price of the box?
- Would any of these graphs have asymptotes?

- What would be the domain and range of your functions?

2.3 Classification of functions

Investigation 11

In this investigation students will learn how to use the horizontal line test to determine if a function is one-to-one or many-to-one.

1 a one-to-one

b one-to-many

c many-to-one

d many-to-many

2 One-to-one and many-to-one

3 Answers will vary, e.g. $y = x$ and $y = x^2$

4 Answers will vary

5 **Conceptual:** How does the horizontal line test help you classify a function?

Answer: If horizontal lines are drawn through the graph of a function and the lines intersect the graph at one point, then the graph represents a one-to-one function. If they intersect the graph at more than one point then the graph represents a many-to-one function.

Investigation 12

Conceptual understanding:

If the horizontal lines drawn through the graph of an onto function intersect the graph of the function in at least one point, this indicates that the domain is mapped to is the set of images of the domain, i.e. the range.

1 Yes

2 No; for example, any graph of the function $y = c, c \in \mathbb{R}$

3 No, because none of the negative real numbers correspond to an element in the domain.

4 $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$

5 a Yes, the range is $\mathbb{R}$.

b No, the range is $\mathbb{R}^+ \cup \{0\}$.

c No, the range is $y \geq -2$.

d Yes, the range is as given, $y \leq 2$.

6 In the graphs of onto functions the horizontal lines cross the graphs at least once, which is not true for functions that are not onto.

7 Change the set of elements that the function is mapped onto to the range of the function.

8 **Conceptual:** Why is the horizontal line test useful for onto functions?

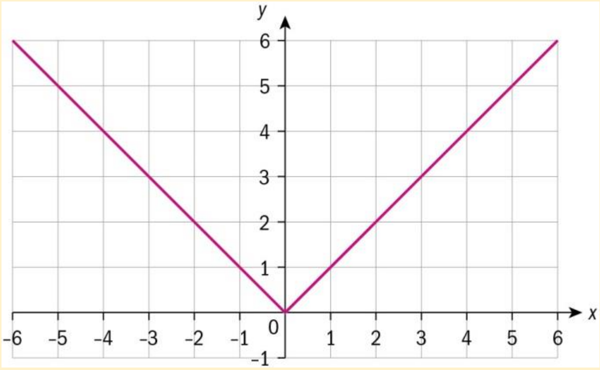
Answer (this is the conceptual understanding): If the horizontal lines drawn through the graph of an onto function intersect the graph of the function in at least one point, this indicates that the domain is mapped to is the set of images of the domain, i.e. the range.

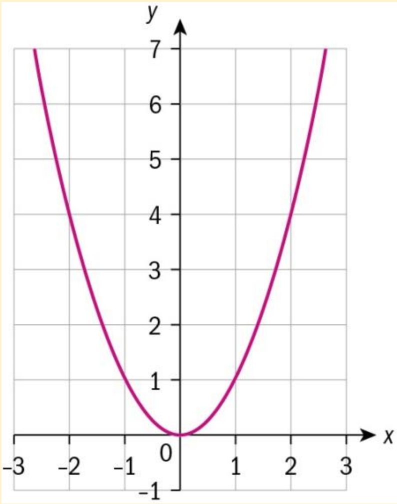
Investigation 13

Conceptual understanding:

If $f(x) = f(-x)$ then the function is even; if $f(-x) = -f(x)$ then the function is odd.

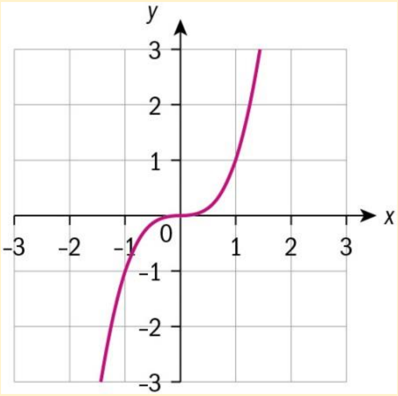
1

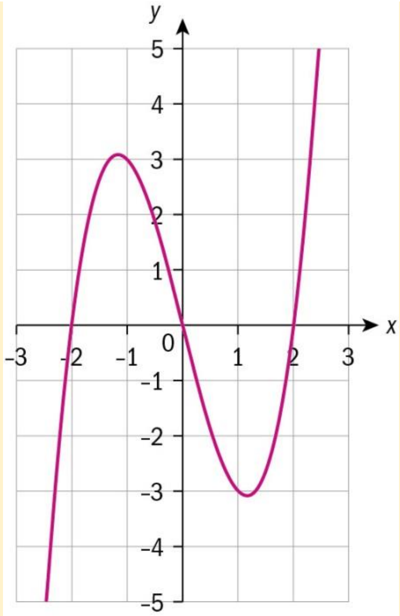




2 $f(a) = f(-a)$

3





4 $f(x) = -f(x)$

5 **Factual:** How do you determine graphically whether a function is even or odd?

Answer: The graph of an even function has reflection symmetry about the y-axis; the graph of an odd function has rotational symmetry of 180° about the origin.

6 **Conceptual:** How do you distinguish whether a function is even or odd?

Answer (this is the conceptual understanding): If $f(x) = f(-x)$ then the function is even; if $f(-x) = -f(x)$ then the function is odd.

Investigation 14

This investigation allows students to determine whether the sum/product of even/odd functions are even/odd.

1 **i** even

ii odd

iii neither

2 **i** even

ii even

iii odd

2.4 Operations with functions

Investigation 15

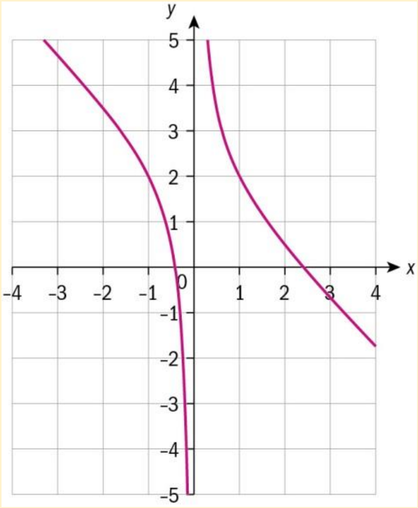
Conceptual understanding:

The domain of a function that consists of more than one function would be the set common to all functions; for the quotient of functions, any values that make the denominator zero would be excluded from the intersection of the sets.

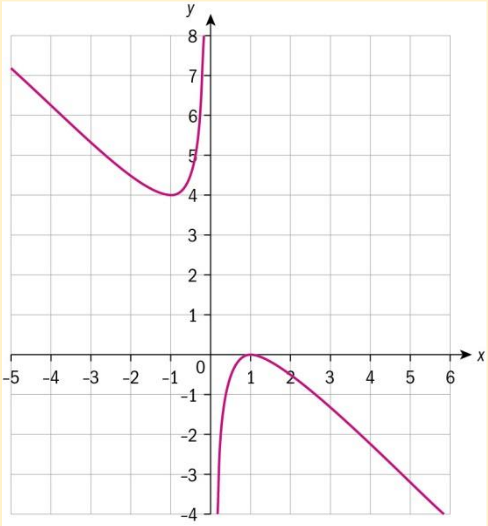
1 $D_f = \{x \in \mathbb{R}\}$; $D_g = \{x \in \mathbb{R}, x \neq 0\}$

2

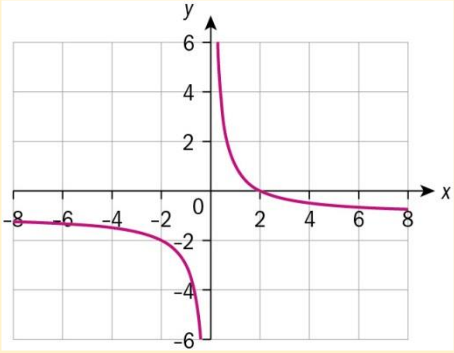
$h(x) = 2 - x + \frac{1}{x}$; $D_h = \mathbb{R} - \{0\}$



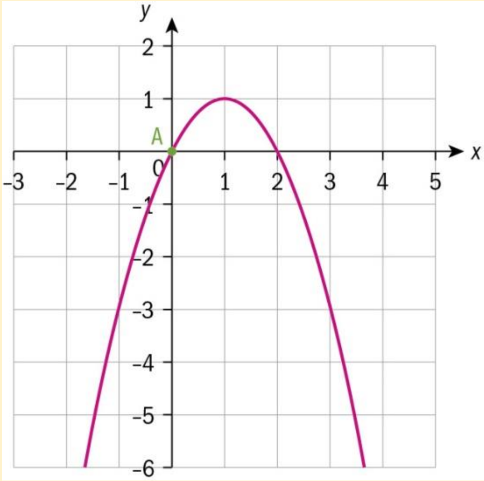
$j(x) = 2 - x - \frac{1}{x}$; $D_h = \{x \in \mathbb{R}, x \neq 0\}$



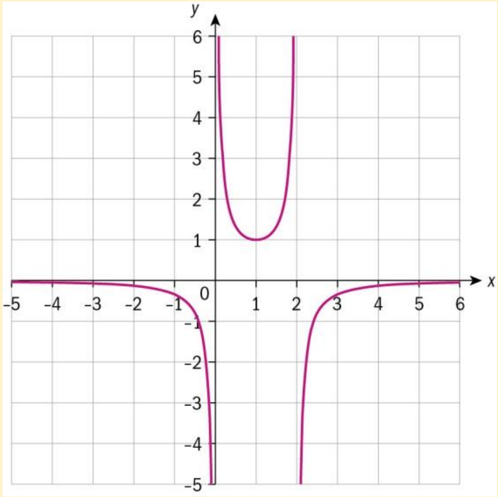
$k(x) = (2 - x) \cdot \frac{1}{x} = \frac{2 - x}{x}$; $D_k = \mathbb{R} - \{0\}$



$p(x) = \dfrac{2-x}{\frac{1}{x}} = x(2-x)$; $D_p = \mathbb{R} - \{0\}$



$q(x) = \dfrac{\frac{1}{x}}{2-x} = \dfrac{1}{x(2-x)}$; $D_q = \mathbb{R} - \{0, 2\}$



3 $D_f \cap D_g$. $D_f \cap D_g = \mathbb{R} - \{2\}$

4 **Conceptual:** How can you obtain the domain of the sum, difference, product or quotient of two or more functions?

Answer (this is the conceptual understanding): The domain of a function that consists of more than one function would be the set that is common to all functions; for the quotient of functions, any values that make the denominator zero would be excluded from the intersection of the sets.

Investigation 16

This investigation allows students to determine if function composition is commutative/associative, and whether compositions of even/odd functions are even/odd.

1 **a** Not commutative

b Associative

2 **a** even

b even

c even

d odd

3 May be neither even nor odd if one of the constituent functions is neither even nor odd.

4 For example, when f and g are both even,

$f \circ g(-x) = f\left(g(-x)\right) = f\left(g(x)\right) = f \circ g(x)$ so $f \circ g$ is even.

TOK

Which do you think is superior: the Bourbaki group analytical approach or the Mandelbrot visual approach to mathematics?

Answer: Nicolas Bourbaki was a pseudonym chosen by eight or nine young mathematicians in the 1930s in France. Their original aim was to write a rigorous textbook in analysis.

The ideas Benoît Mandelbrot, who died in 2018, gave a new, visual way to think about the world. The formal techniques of theorems and proofs didn't appeal to him. His approach to mathematics was highly visual and highly intuitive.

Allow students to research each and write a response.

Investigation 17

In this investigation students will discover that a composite function does not have a unique decomposition.

1 $f(x) = x^2 + 3; g(x) = x - 1$

2 Change the function into vertex form.

3 $h(x) = x^2 + 2.75; k(x) = x - 0.5$

4 $h(k(x)) = x^2 - x + 3 = (x - 0.5)^2 + 2.75$

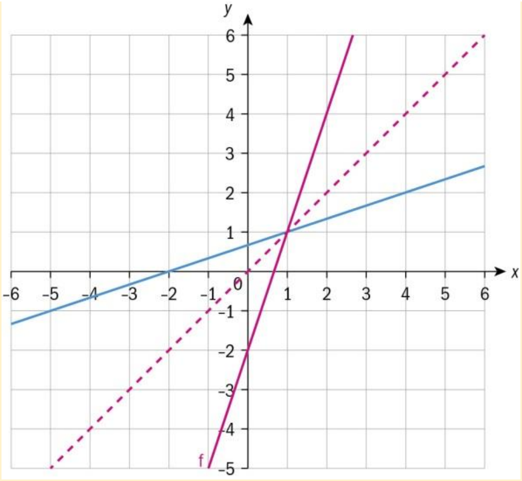
5 No.

Investigation 18

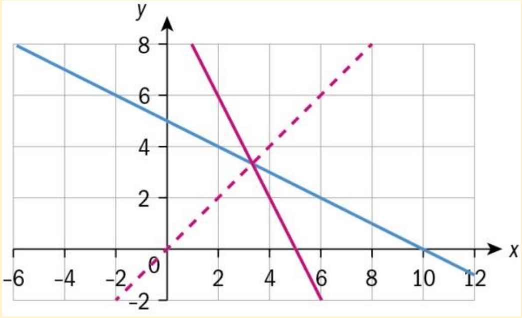
Conceptual understanding:

Inverse functions have reflection symmetry about the line y=x, since the reflection line $y = x$ bisects the line segment joining any two corresponding points between the graph of the function and its inverse.

1 $y = x$



2



3 The gradient of the segment is −1. Since the gradient of $y = x$ is 1, the lines are perpendicular, and $y = x$ bisects the line segment, ie the reflection line is the perpendicular bisector of the line joining corresponding points on the function and its inverse. The distances therefore from where the line intersects $y = x$ to the function and its inverse are equal.

4 **Conceptual:** How would you justify graphically that two functions are inverses of each other?

Answer (this is the conceptual understanding): Inverse functions have reflection symmetry about the line y=x, since the reflection line $y = x$ bisects the line segment joining any two corresponding points between the graph of the function and its inverse.

5 Each of these functions are linear functions, and hence are one-to-one.

Investigation 19

Conceptual understanding:

Only functions both one-to-one and onto have inverse functions, since a many-to-one function has a one-to-many inverse.

1 Many-to-one

2 Yes

3 Its graph does not pass the vertical line test.

4 $x \geq 0$

5 $D_f = R^- \cup \{0\}, R_f = R^- \cup \{0\}; f(x) = \dfrac{1}{x-2} + 3, x \neq 2$

6 $D_f = R^- \cup \{0\}, R_f = R^- \cup \{0\}; D_{f^{-1}} = R^- \cup \{0\}, R_{f^{-1}} = R^- \cup \{0\}$;

7 The vertex.

8 **Conceptual:** What kind of functions (one-to-one, many-to-one, onto) have inverses that are also functions? Why?

Answer (this is the conceptual understanding): Only functions both one-to-one and onto have inverse functions, since a many-to-one function has a one-to-many inverse.

Investigation 20

This investigation allows students to see some general forms of self-inverse functions.

1 Yes.

2 Yes.

3 Yes.

4 Answers will vary.

2.5 Function transformations

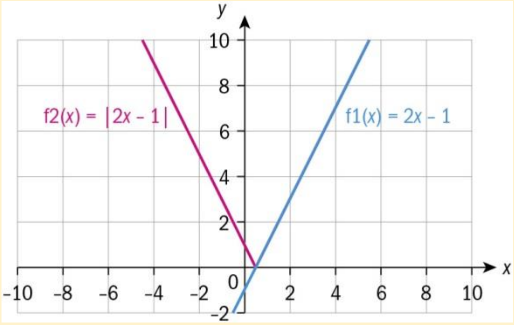
Investigation 21

Conceptual understandings:

The graph of the *absolute value of a function* remains unchanged for non-negative *y*-values of the function and reflected in the *x*-axis for negative *y*-values since the range of the function is the set of positive real numbers.

The graph of the *function of the absolute value* remains unchanged for non-negative *x*-values of the function, and for negative *x*-values; the graph for its non-negative *x*-values is reflected in the *y*-axis.

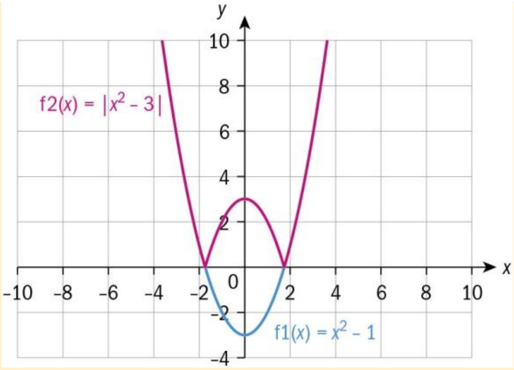
1 a



b i D=Reals; R=Reals

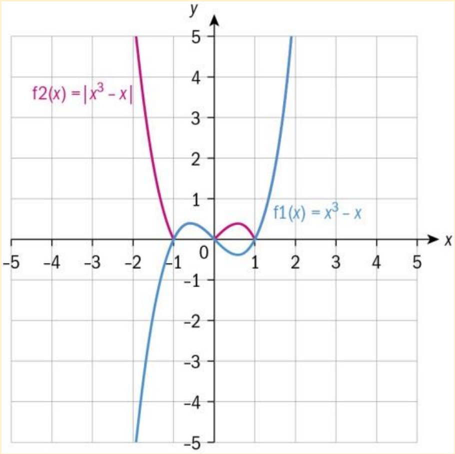
ii D=Reals; R: $y \geq 0$

2 i



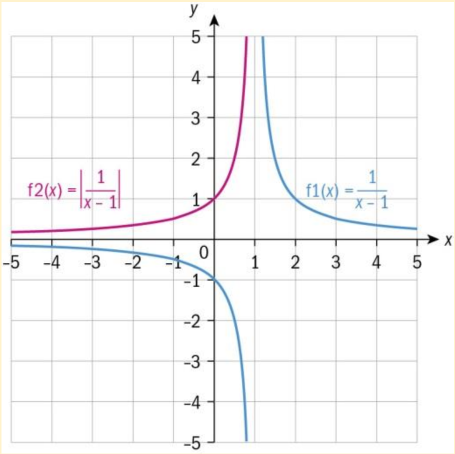
a D=Reals; R: $y \geq 3$ **b** D=Reals; R: $y \geq 0$

ii



a D=Reals, R=Reals **b** D=Reals, R: $y \geq 0$

iii

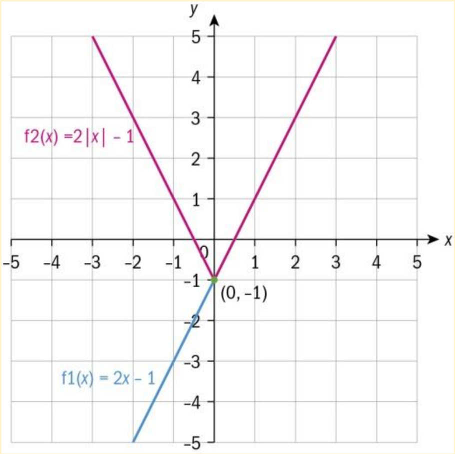


a D: $x \neq 1$; R: $y \neq 0$; **b** D: $x \neq 1$; R: $y \geq 0$

3 **Conceptual:** How does the graph of $y = |f(x)|$ change compared to the graph of $y = f(x)$ over the range of $y = f(x)$?

Answer (this is a conceptual understanding): The graph of the *absolute value of a function* remains unchanged for non-negative y-values of the function and reflected in the x-axis for negative y-values since the range of the function is the set of positive real numbers.

4

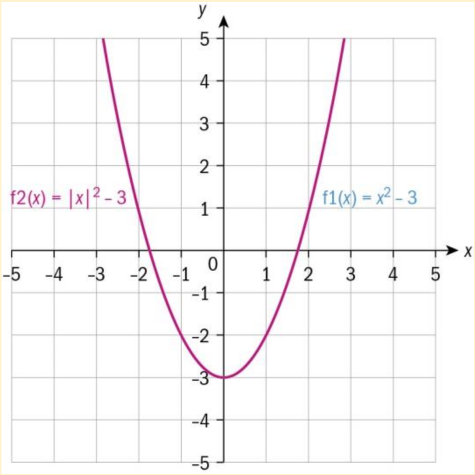


b **i** D=Reals; R=Reals

ii D=Reals, R: y≥-1

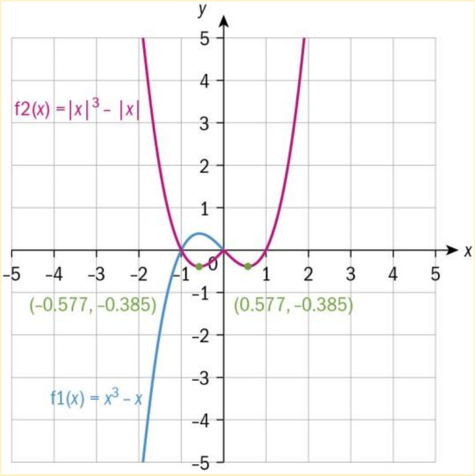
5

i



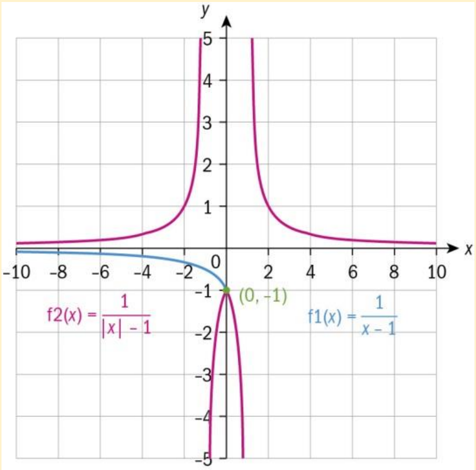
a D=Reals; R: y≥-3 **b** D=Reals; R:y≥-3

ii



a D: Reals; R=Reals **b** D: Reals; R: y≥-0.385(to 3sf)

iii



a D: $x \neq 1$; R: $y \neq 0$ **b** D: $x \neq \pm 1$, $y > 0$ or $y \geq -1$

6 **Conceptual:** How does the graph of $y = f(|x|)$ change compared to the graph of $y = f(x)$ over the domain of $y = f(x)$?

Answer (this is a conceptual understanding): The graph of the *function of the absolute value* remains unchanged for non-negative x-values of the function, and for negative x-values; the graph for its non-negative x-values is reflected in the y-axis.

TOK

When students see a familiar equation with a transformation, they will often get a “gut feeling” about what the function looks like.

Respond to this question. Is intuition helpful or harmful in mathematics?

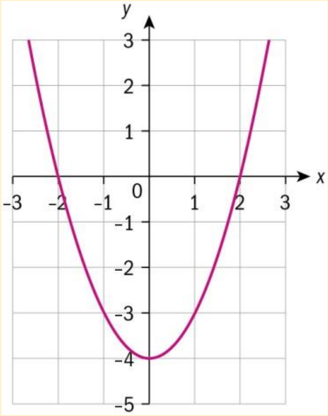
Answer: Questions that a teacher might like to pose include:

- Is it a bad idea to try to solve problems using our “gut feeling”, intuition, or use reason and evidence to make a decision?
- Where does that gut feeling come from?
- When has it helped you?
- When has it misled you?

Investigation 22

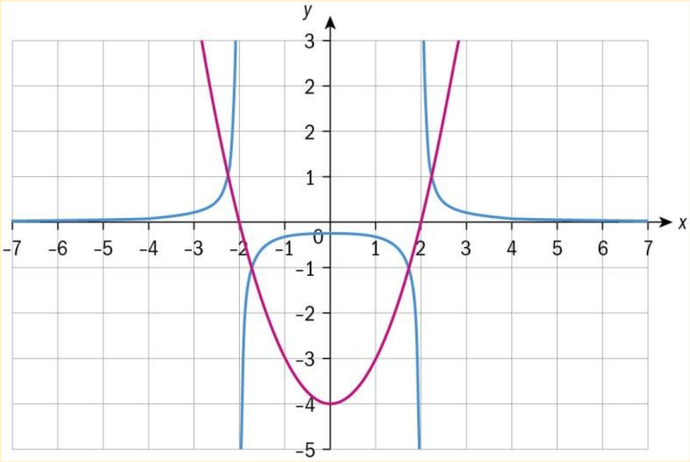
Students will discover the steps for transforming the graph of $y = f(x)$ into the graphs of $y = |f(x)|$ and $y = f(|x|)$.

1



D=Reals; R:y≥4

2



D: x≠±2, R: y≠0

3 At the x-intercepts of f, g has asymptotes.

4 The y-intercept of g is the reciprocal of the y-intercept of f.

5 The minimum of f becomes a maximum of g, and vice-versa.

6 **a** Where $f(x) > 0$,$\frac{1}{f(x)} > 0$, and where $f(x) < 0$, $\frac{1}{f(x)} < 0$.

b When $f(x)$ approaches $\pm\infty$, $\frac{1}{f(x)}$ approaches 0. When $f(x)$ approaches 0, $\frac{1}{f(x)}$ approaches ∞.

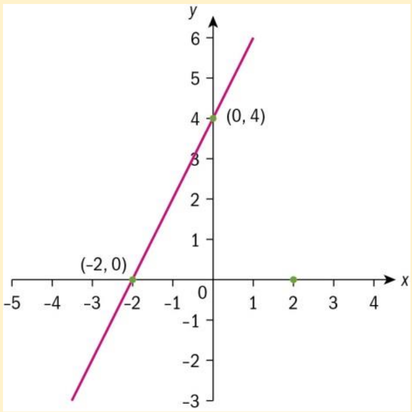
7 To transform $y = f(x)$ to $y = |f(x)|$, the graph is unchanged for y≥0, and reflected in the x-axis for y≤0.

To transform $y = f(x)$ to $y = f(|x|)$, the graph is unchanged where x≥0, and for x<0, the graph for x≥0 is reflected in the y-axis.

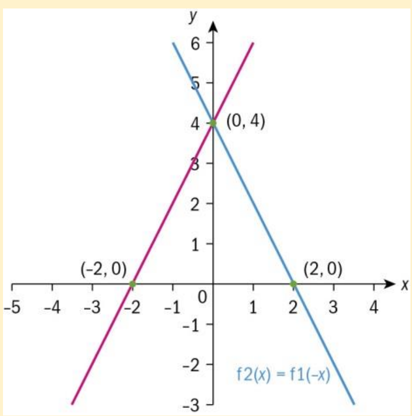
Investigation 23

Students will discover the symmetrical properties of the graphs of $y = f(x)$ and $y = -f(x)$.

1

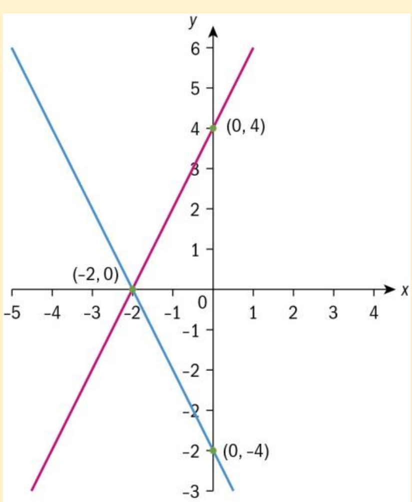


2 $f(x) = -2x - 4$



3 The graphs are symmetrical about the y-axis.

4 $-f(x) = -2x - 4$



5 The graphs are symmetrical about the x-axis.

TOK

Is mathematics independent of culture?

Answer: Some believe that mathematics is its own language and needs no reference to applications.

Would mathematics exist without context?

If so, would this make mathematics an art form?

The culture of mathematics has been derived from many different cultures over millennia and has been transferred between different nationalities and religions.

Is mathematics a culture in itself?

Investigation 24

Students' graphs.

Investigation 25

Conceptual understanding:

The order in which transformations on a graph of a function are carried out influence the final result.

1 $g(x) = \frac{1}{3x}$; the graph is compressed by a factor of $\frac{1}{3}$, ie it moves closer to the y-axis.

2 $h(x) = 2g(x) = \frac{2}{3x}$

3 $k(x) = h(x-2) = \frac{2}{3(x-2)}$

4 $p(x) = k(x) - 1 = \frac{2}{3(x-2)} - 1$

5 $p(x) = \frac{-3x+8}{3x-6}; x \neq 2$

6 No.

7 The dilations are done first in any order, then the translations are done in any order.

8 $g(x) = f(x-6)$

9 $h(x)=3x$

10 **Conceptual:** How is the order in which transformations on a function are performed important to the outcome?

Answer (this is the conceptual understanding): The order in which transformations on a graph of a function are carried out influence the final result.

11 Begin with the argument and work your way outwards.

TOK

Do you think that mathematics is just the manipulation of symbols under a set of rules?

Answer: Questions that can be used to stimulate a discussion or a blog post about the nature of mathematics include:

- What is mathematics?
- Is it all shared knowledge?
- How can you develop personal knowledge in mathematics?
- Do you just use formulas to reason and evaluate?
- Does faith ever have a place in mathematics?

Reflect: How do the transformations change if you do the horizontal shift before the horizontal dilation?

Answer: In Example 40, the order of transformations meant that the first two transformations were $|x| \to |2x| \to \left|2\left(x+\frac{1}{2}\right)\right| = |2x+1|$

Instead, doing the horizontal shift first, followed by the horizontal dilation would mean the first two translations were $|x| \to \left|x+\frac{1}{2}\right| \to \left|2\left(x+\frac{1}{2}\right)\right| = |2x+1|$, so interchanging the order of these two transformations gives the same result.

Modelling and investigation activity: To Infinity and .... !

Approaches to Learning: Thinking Skills, Communication, Research, Collaboration

Exploration Criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB Topic: Linking different areas

In Chapter 4 students will consider the crucial role that infinity plays in calculus where the idea of an infinitely small quantity is used. It has also already come up in chapter 1 and in this chapter. Rather than simply preparing students for an exploration, this RLT is designed to address the curiosity that students often feel regarding infinity as a concept. There are obviously some clear Theory of Knowledge ideas that can be addressed as well. It is not actually expected that this will lead to an exploration topic because often explorations based on something like this will not be successful due to them being low scoring in Criterion E: Use of mathematics.

However, the RLT may prompt students to think about researching and finding appropriate sites and books. The suggestion in the extension box of a presentation to the class will hopefully encourage students to communicate complex ideas in a way that other students in their class will understand. Conducting research, choosing a demonstrable area of interest and seeing the links between areas of mathematics all contribute towards Criterion C: Personal engagement. Students could be given a homework to research and prepare the presentation and be prepared to present for 5 minutes per group in the next lesson.

What is infinity?

In these early discussions it is important to address the misconception that infinity is an ordinary number. It is not!

Discussions may also lead towards thinking about different kinds of infinity rather than there being one single entity that is infinity.

As will be seen, the understanding and study of the concept of infinity has led to advances in mathematics and science.

It is also very intriguing and a little disturbing at times! (The ancient Greeks, for example, viewed it with suspicion and hostility and some refused to recognize it!)

As **extension** work, you could encourage students to research the history of the concept of infinity.

Thinking about the concept of infinity is quite complicated and produces some quite challenging questions and some surprising results and paradoxes (and a few headaches)! The introduction of the concept infinity is credited to a philosopher named Zeno of Elea, who lived from 490 B.C. to 430 B.C. Since that time mathematicians, physicists, theologians, philosophers and even artists have grappled with the concept of infinity.

Infinity is a concept that students will encounter many times in this course.

They will meet infinity again when they study Geometric sequences and calculus (differentiation and then integration) in future chapters. Infinity plays a crucial role in calculus, where the idea of an infinitely small quantity is used.

Other examples of where students have met the concept of infinity could include:

Lines (with infinite length)

Sets of numbers (real, rational, integers, etc)

Irrational numbers (pi, e, etc)

Non-terminating rational numbers

Recurrence relations

Sequences and Series

Limits in differentiation

Sums of areas under a curve leading to integration

Art - perspective point

If appropriate, you could ask students to give a short presentation on one of the examples they give.

Let's think further about infinity

The following three games each look at infinity in a slightly different way.

Before each game starts, you should randomly select the order of players in the class.

Game 1: The winner is the person who names the biggest positive natural number.

The last player will win.

Game 2: The winner is the person who names the closest rational number to 0.

The last player will win.

Game 3: The winner is the person who names the closest real number to 1.

The last player will win.

To prompt discussion, you could ask:

What if, in these games, no one goes last?

What if you never stop playing and keep on going around the class?

If no one went last and they never stop playing the games, no one would ever win and the games would go on forever.

In game 1, the natural numbers chosen can be infinitely large. They can all be listed. However, this is clearly impractical. If you counted forever, you'd be sure not to miss any out. Therefore, they are countably infinite.

In game 2, the numbers need to be infinitely small. Students are only allowed rational numbers (those that can be written as a fraction). There are seemingly more rational numbers than natural numbers but as it is still possible to count them all and to map them on to the natural numbers, there are actually the same number!

You could use this table to convince students of this. Every possible number *p*/*q* is represented (it is on the *p*th row and *q*th column).

1/1	1	1/2	2	1/3	6	1/4	7	1/5	15	1/6	16	…
2/1	3	2/2	5	2/3	8	2/4	14	2/5	17	…		
3/1	4	3/2	9	3/3	13	3/4	18	…				
4/1	10	4/2	12	4/3	19	…						
5/1	11	5/2	20	…								
6/1	21	…										
…												

You can count these numbers, not by counting along the rows (as you would never finish the first row), but rather by following the order numbers in the second columns. In this way you have counted every rational once (as long as you ignore those rational numbers that are not in their simplest form).

In game 3, the set of numbers concerns the real numbers. The real numbers are not actually countable (they are uncountably infinite). The proof of this involves Cantors Diagonal Proof which is one of the possible presentations in the **extension** box.

Conclusion

The answer is no. Two sets that contain an infinite number of numbers are not necessarily the same size.

One infinity can be larger than another because one infinity can be countable, whereas another infinity may not be countable.

Extension presentation

Students could also research another concept that is not in the list, if preferred.

Students could work individually, or you could pair/group students with shared interests.

You could provide advice regarding good research techniques, using the internet, recording sources, going to the library, etc.

The idea here is that students are grappling with some complicated concepts and thinking about how best to present these to a class of their peers - this can be challenging but perhaps is a timely opportunity to suggest to students that they will be submitting an IA that has an 'audience of their peers'.

Make sure that students use appropriate language in their presentation.

Researching the concepts of infinity is a good introduction to looking at the concept of infinity and limits in Chapter 4 on calculus.

3 Expanding the number system: complex numbers

Essential understandings

Number and algebra allow us to represent patterns, show equivalencies and make generalizations which enable us to model real-world situations. Algebra is an abstraction of numerical concepts and employs variables which allow us to solve mathematical problems.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Different representations of numbers enable equivalent quantities to be compared and used in calculations with ease to an appropriate degree of accuracy.
- Numbers and formulae can appear in different, but equivalent, forms, or representations, which can help us to establish identities.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Equivalent representations of quadratic functions can reveal different characteristics of the same relationship.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
Completing the square converts a quadratic expression to a square plus another term and the process can be visualized by representing the terms in the expression as areas and finding the 'missing' area that would make a square	Investigation 1
The quadratic formula generalizes the solutions to quadratic equations by utilizing completing the square.	Investigation 2
The discriminant of a quadratic equation can be used to determine the nature of the roots and the number of *x*-intercepts of the corresponding quadratic function.	Investigation 3
Mathematicians use the imaginary number i to represent the square root of −1 which allows for undefined solutions to quadratic equations to be expressed.	Investigation 4

Every complex number may be represented by a point in the complex plane, and the modulus expresses the distance between the point representing the complex number z and the origin.	Investigation 5
Complex number addition and multiplication follow the rules of algebra where a complex number may be treated in a similar way to a polynomial: multiples of i are grouped together, and constant terms are grouped together.	Investigation 6
Complex number multiplication follows the rules of algebra where a complex number may be treated as a polynomial after reducing the powers of the imaginary number i.	Investigation 7
The complex conjugate of a complex number displays equal real parts and an imaginary part equal in magnitude but opposite in sign.	Investigation 8
The powers of imaginary numbers and complex numbers may be generalized to a repeating linear sequence with period four.	Investigation 9
All the even power graphs (including $n = 0$) are symmetrical about the y-axis (these are even functions) and all the odd power graphs (including $n = 1$) have rotational symmetry, order 2, about the origin (these are odd functions).	Investigation 10
The parameters of a cubic function alter the shape, turning points and intercepts of the graph of the function.	Investigation 11
Any number can be expressed as the product of prime factors. Any polynomial may be expressed as the product of one or more simple linear polynomial factors called prime factors.	Investigation 12
Complex zeros of polynomial functions occur in conjugate pairs.	Investigation 13
The sum of the zeros of a cubic polynomial equals the opposite ratio of the quadratic coefficient and the cubic coefficient, and the product of the roots equals the opposite ratio of the constant term and the cubic coefficient.	Investigation 14
The sum of the zeros of a fourth-degree polynomial equals the opposite ratio of the cubic coefficient and the quartic coefficient, and the product of the roots equals the ratio of the constant term and the quartic coefficient.	Investigation 15
Solutions to systems of two linear equations with two unknowns can produce: a unique solution when the lines intersect at one point; no solution when the lines are parallel; and infinitely many solutions when the lines coincide.	Investigation 16
Solutions to systems of three linear equations with three unknowns can produce a unique solution, no solution and infinitely many solutions.	Investigation 17

Syllabus sections covered in this chapter:

- SL2.6
- SL2.7
- AHL1.12
- AHL1.14
- AHL1.16
- AHL2.12

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and the development of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 147: Solving quadratic equations, finding intersection points of graphs, solving linear and quadratic inequalities, solving linear simultaneous equations	Page 155: Example 5 Page 169: Example 17 Page 178: Example 21 Page 183: Example 26 Page 208: Example 43	Page 158: Example 8 Page 164: Example 12 Page 165: Example 13 Page 167: Example 14 Page 169: Example 17 Page 173: Example 19 Page 185: Example 27 Page 188: Example 29 Page 197: Example 34 Page 198: Example 35 Page 201 Page 209: Example 44	Pages 160, 174, 184, 194, 200, 210

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 211	Page 214	N/A

3.1 Quadratic equations and inequalities

International-mindedness
The Babylonians (2000-1600BC) used quadratics to find the area of fields for agriculture and taxation.

Investigation 1

Conceptual understanding:

Completing the square converts a quadratic expression to a square plus another term and the process can be visualized by representing the terms in the expression as areas and finding the 'missing' area that would make a square.

1 $c = 9$; $x^2 + 4x + 4 = (x+2)^2$

2 $c = 25$; $x^2 + 10x + 25 = (x+5)^2$

3 $c = \frac{9}{4}$; $x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$

4 $c = \left(\frac{b}{2}\right)^2$; $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

5 $\left(\frac{b}{2}\right)^2$

6 **Factual:** What are the steps of the completing the square method?

Answer: When quadratic term is equal to 1 then the quadratic and linear term need a constant term that is a square of a half of the linear coefficient, to make a perfect square of a binomial linear expression.

7 **Conceptual:** How does the area model help you understand the process of completing the square and why the process is called completing the square?

Answer (this is the conceptual understanding): Completing the square converts a quadratic expression to a square plus another term and the process can be visualized by representing the terms in the expression as areas and finding the 'missing' area that would make a square.

Investigation 2

Conceptual understanding:

The quadratic formula generalizes the solutions to quadratic equations by utilizing completing the square.

Mathematical working	Explanation
$ax^2+bx+c=0$	
1 $a^2x^2+abx+ac=0$	**1** Multiply both sides of the equation by *a*.
2 $a^2x^2+abx=-ac$	**2** Subtract *ac* from both sides of the equation.
3 $a^2x^2+2\cdot ax\cdot\frac{b}{2}+\left(\frac{b}{2}\right)^2=-ac+\left(\frac{b}{2}\right)^2$	**3** Complete the square on the left-hand side, and add the same constant term to the right-hand side.
4 $\left(ax+\frac{b}{2}\right)^2=\frac{b^2}{4}-ac$	**4** Factorise the left-hand side.
5 $\left(ax+\frac{b}{2}\right)^2=\frac{b^2-4ac}{4}$	**5** Put the right hand side to the common denominator.
6 $ax+\frac{b}{2}=\pm\sqrt{\frac{b^2-4ac}{4}}$	**6** Take the square root of both sides.
7 $ax=-\frac{b}{2}\pm\frac{\sqrt{b^2-4ac}}{2}$	**7** Subtract $\frac{b}{2}$ from both sides.
8 $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	**8** Divide by *a* to find *x*.

Factual: What do *a*, *b*, and *c* stand for in your formula in line 8?

Answer: The coefficients of the original quadratic equation, $ax^2+bx+c=0, a, b, c\in\mathbb{R}, a\neq 0$

Conceptual: How does the quadratic formula generalize the solutions to quadratic equations?

Answer (this is the conceptual understanding): The quadratic formula generalizes the solutions to quadratic equations by utilizing completing the square.

TOK

How can you deal with the ethical dilemma of using mathematics to cause harm, such as plotting the course of a missile?

Answer: Questions that might be used include:
Should mathematics be used to take lives?
What are the decisions to be made when developing the accuracy and stealth of weapons such as drones?
Consider the role of the scientists developing the atomic bomb.

Investigation 3

Conceptual understanding:

The discriminant of a quadratic equation can be used to determine the nature of the roots and the number of x-intercepts of the corresponding quadratic function.

1 a $x = -2$ or $x = 5$

b No real solutions

c $x = 6$

d $x = \dfrac{5}{2}$

e $x = -2$ or $x = 5$

f No real solutions

2 **Factual:** How many real solutions can different quadratic equations have?

Answer: Some quadratic equations have two real solutions, some have one real solution, and other have no real solutions.

3 **Conceptual:** By considering the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, explain why quadratic equations have different numbers of solutions.

Answer: The equation $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ contains the square root $\sqrt{b^2 - 4ac}$, which only yields a real value if $b^2 - 4ac \geq 0$.

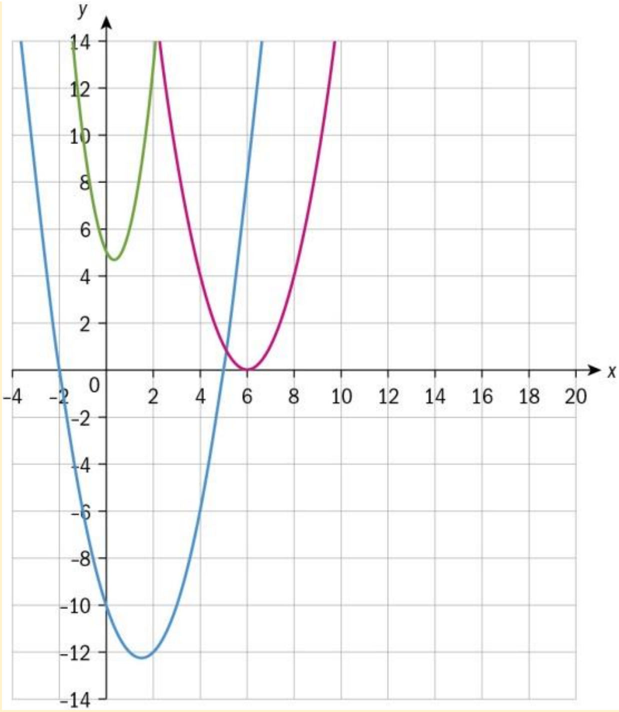
Students may be able to go further to ascertain that

if $b^2 - 4ac > 0$, equation has two real solutions

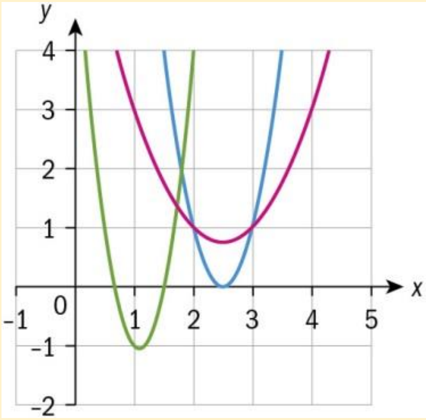
if $b^2 - 4ac = 0$, equation has only one real solution

if $b^2 - 4ac < 0$, then the square root is undefined and the equation has no real solutions.

4 Graphs of functions in **a, b, c** are shown below:



Graphs of functions in **d, e, f** are shown below:



5 **Factual:** What do you notice about the number of zeros of each graph, and the number of solutions to the quadratic equation?

Answer: The number of solutions corresponds to the number of zeros (points of intersection with the *x*-axis) of the quadratic function.

6 **Factual:** What does the graph of a quadratic equation tell you about the value of the discriminant?

Answer: If the discriminant is:

greater than 0, the parabola will have two points of intersection with the *x*-axis

equal to 0 the parabola will have one point at which the *x*-axis touches the parabola (the *x*-axis will be a tangent to the parabola at this point)

less than 0, there are no points of intersection between the parabola and the *x*-axis.

7 **Conceptual:** What can the discriminant be used for?

Answer (this is the conceptual understanding): The discriminant of a quadratic equation can be used to determine the nature of the roots and the number of *x*-intercepts of the corresponding quadratic function.

International-mindedness

Frenchman Nicole Oresme was one of the first mathematicians to consider the concept of functions in the 14th century. The term "function" was introduced by the German mathematician Gottfried Wilhelm Leibniz in the 17th century and the $f(x)$ notation was coined by Swiss Leonard Euler in the 18th century.

Answer: This international-mindedness box and the next two all relate to the TOK question in the box below.

TOK

We have seen the involvement of several nationalities in the development of quadratics.
To what extent do you believe that mathematics is a product of human social collaboration?

Answer: A 500-word response on paper or in a blog is a suitable way of answering this ToK question.

Answers might include that mathematics allows humans to pass on knowledge from one another, across national, cultural and religious borders. The social communication required for the symbology and language to develop is a form of communication where people interact with one another whilst passing on the knowledge of mathematics.

A counter claim might be that many mathematicians (e.g. Andrew Wiles) work best in isolation.

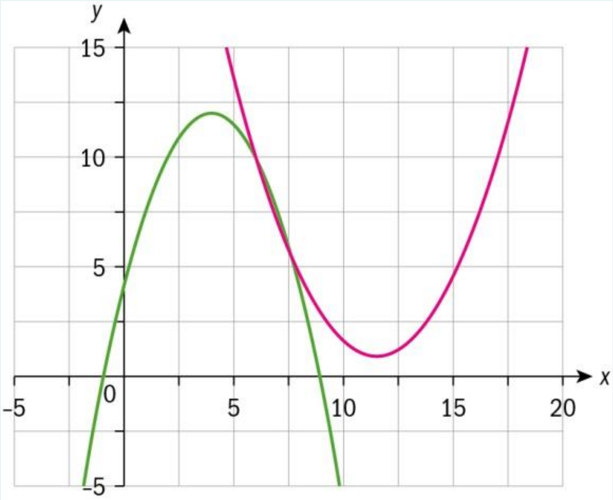
Developing inquiry skills

Return to the opening problem. Can you use a quadratic equation to model the roller coaster?

Answer: You could use a number of different quadratic functions, each with restricted domain, in order to model the path of the roller coaster. This could be interpreted as a piecewise quadratic function.

For example, the following piecewise quadratic function could model the first two vertices of the track. One graph has a maximum and the other has a minimum:

$$y = \begin{cases} -0.5(x-4)^2 + 12, & 0 \le x \le 7.5 \\ 0.3(x-11.5)^2, & 7.5 < x < 16 \end{cases}$$



3.2 Complex numbers, modulus, operations with complex numbers

TOK

Descartes showed that geometric problems could be solved algebraically and vice versa. What does this tell us about mathematical representation and mathematical knowledge?

Answer: Questions to start a discussion in class might include:
Is there a difference in accuracy between geometrical and algebraic representation?
Which forms of representation stay in your memory for longer? Formulas, diagrams, colours? Why?

Investigation 4

Conceptual understanding:

Mathematicians use the imaginary number i to represent the square root of −1 which allows for undefined solutions to quadratic equations to be expressed.

1 There are no real values that satisfy this equation, since there is a negative number under the square root.

2 $x = \pm\sqrt{-4} = \pm\sqrt{-1 \cdot 4} = \pm\sqrt{-1} \cdot \sqrt{4} = \pm 2\mathrm{i}$

3 $x = \dfrac{2 \pm \sqrt{4-20}}{2} = \dfrac{2 \pm \sqrt{-16}}{2} = \dfrac{2 \pm 4\mathrm{i}}{2} = 1 \pm 2\mathrm{i}$

4 **Conceptual:** What does the imaginary number i represent and how is it used in solving quadratics?

Answer (this is the conceptual understanding): Mathematicians use the imaginary number i to represent the square root of −1 which allows for undefined solutions to quadratic equations to be expressed.

International-mindedness

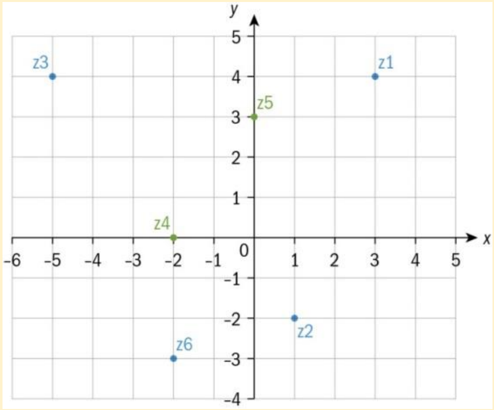
Greek mathematician Heron discussed the root of a negative in the first century BC.

Investigation 5

Conceptual understanding:

Every complex number may be represented by a point in the complex plane, and the modulus expresses the distance between the point representing the complex number z and the origin.

1



2 5, $\sqrt{5}$, $\sqrt{41}$, 2, 3, $\sqrt{13}$

3 **Factual:** Can you write down a formula for finding the modulus, $|z|$, of a complex number $z = x + \mathrm{i}y$?

Answer: $|z| = \sqrt{x^2 + y^2}$

4 **Factual:** What do you notice about the modulus of a complex number?

Answer: The modulus of a complex is always a real, nonnegative number because it represents a distance.

5 **Conceptual:** What does the modulus of a complex number tell you about that number?

Answer (this is the conceptual understanding): Every complex number may be represented by a point in the complex plane, and the modulus expresses the distance between the point representing the complex number *z* and the origin.

Investigation 6

Conceptual understanding:

Complex number addition and multiplication follow the rules of algebra where a complex number may be treated in a similar way to a polynomial: multiples of i are grouped together, and constant terms are grouped together.

z_1	z_2	$z_1 + z_2$	m	$m \cdot z_1$	n	$n \cdot z_2$		$m \cdot z_1 + n \cdot z_2$
$z_1 = 2 + \mathrm{i}$	$z_2 = 3 - 2\mathrm{i}$	$5 - \mathrm{i}$	2	$4 + 2\mathrm{i}$	-3	$-9 + 6\mathrm{i}$		$-5 + 8\mathrm{i}$

(Students should then add their own examples in the subsequent rows).

1 **Factual:** How do you add two complex numbers?

Answer: Two complex numbers are added by adding their real parts and adding their imaginary parts.

2 **Factual:** How do you subtract two complex numbers?

Answer: Two complex numbers are subtracted by subtracting their real parts and subtracting their imaginary parts.

3 **Conceptual:** Do we need two separate operations for adding and subtracting complex numbers?

Answer: No; Subtraction can be seen as adding the additive inverse (the negative number). In both cases, real parts are added together, and imaginary part are added together.

4 **Factual:** How do you multiply (or divide) a complex number by a real number?

Answer: You multiply (or divide) both real part, and imaginary part, by that real number.

5 **Conceptual:** How are algebraic operations with complex numbers similar to algebraic operations with polynomials?

Answer (this is the conceptual understanding): Complex number addition and multiplication follow the rules of algebra where a complex number may be treated in a similar way to a polynomial, where multiples of i are grouped together, and constant terms are grouped together.

International-mindedness

How do you use the Babylonian method of multiplication? Try 36×14.

Answer: Using a table of squares and the formula $xy = \left\lfloor (x+y)^2 - (x-y)^2 \right\rfloor \div 4$.

$$36 \times 14 = \left\lfloor (36+14)^2 - (36-14)^2 \right\rfloor \div 4$$

$$36 \times 14 = \left\lfloor (50)^2 - (22)^2 \right\rfloor \div 4$$

$$36 \times 14 = \frac{2016}{5} = 504$$

Investigation 7

Conceptual understanding:

Complex number multiplication follows the rules of algebra where a complex number may be treated as a polynomial after reducing the powers of the imaginary number i.

z_1	z_2	$z_1 \cdot z_2$
$z_1 = 2 + \mathrm{i}$	$z_2 = 1 + 3\mathrm{i}$	$(2 + \mathrm{i}) \cdot (1 + 3\mathrm{i}) = 2 + \mathrm{i} + 6\mathrm{i} + 3\underbrace{\mathrm{i}^2}_{-1} = -1 + 7\mathrm{i}$
$z_1 = 1 - 2\mathrm{i}$	$z_2 = 3 + \mathrm{i}$	$(1 - 2\mathrm{i}) \cdot (3 + \mathrm{i}) = 3 + \mathrm{i} - 6\mathrm{i} - 2\underbrace{\mathrm{i}^2}_{-1} = 5 - 5\mathrm{i}$
$z_1 = 3 + 2\mathrm{i}$	$z_2 = 4 - 5\mathrm{i}$	$(3 + 2\mathrm{i}) \cdot (4 - 5\mathrm{i}) = 12 - 15\mathrm{i} + 8\mathrm{i} - 10\underbrace{\mathrm{i}^2}_{-1} = 22 - 7\mathrm{i}$
$z_1 = a + b\mathrm{i}$	$z_2 = 3 + \mathrm{i}$	$(a + b\mathrm{i}) \cdot (3 + \mathrm{i}) = 3a + a\mathrm{i} + 3b\mathrm{i} + b\underbrace{\mathrm{i}^2}_{-1} = (3a - b) + (a + 3b)\mathrm{i}$
$z_1 = 3 + 2\mathrm{i}$	$z_2 = c + d\mathrm{i}$	$(3 + 2\mathrm{i}) \cdot (c + d\mathrm{i}) = 3c + 3d\mathrm{i} + 2c\mathrm{i} + 2d\underbrace{\mathrm{i}^2}_{-1} = (3c - 2d) + (3d + 2c)\mathrm{i}$
$z_1 = a + b\mathrm{i}$	$z_2 = c + d\mathrm{i}$	$(a + b\mathrm{i}) \cdot (c + d\mathrm{i}) = ac + ad\mathrm{i} + bc\mathrm{i} + bd\underbrace{\mathrm{i}^2}_{-1} = (ac - bd) + (ad + bc)\mathrm{i}$

1 **Factual:** Using words, express how the real part of $z_1 \times z_2$ is related to the real and imaginary parts of both z_1 and z_2.

Answer: Real part of $z_1 \cdot z_2$ is the difference between

i the product of the real parts of z_1 and z_2, and

ii the product of the imaginary parts of z_1 and z_2.

2 **Factual:** In a similar way, express how the imaginary part of $z_1 \cdot z_2$ is related to the real and imaginary parts of both z_1 and z_2.

Answer: Imaginary part of $z_1 \times z_2$ is the sum of

i the product of the real part of z_1 with the imaginary part of z_2,

ii the product of the imaginary part of z_1 with the real part of z_2.

3 **Conceptual:** How would you compare multiplication of two complex numbers to multiplication of two algebraic linear factors?

Answer (this is the conceptual understanding): Complex number multiplication follows the rules of algebra where a complex number may be treated as a polynomial after reducing the powers of the imaginary number i.

Investigation 8

Conceptual understanding

The complex conjugate of a complex number displays equal real parts and an imaginary part equal in magnitude but opposite in sign.

z	$\lvert z\rvert$	z^*	$z \cdot z^*$
$z = -1 + \mathrm{i}$	$\lvert z\rvert = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$	$z = -1 - \mathrm{i}$	$z \cdot z^* = 2$
$z = 3 - 4\mathrm{i}$	$\lvert z\rvert = \sqrt{3^2 + (-4)^2} = 5$	$z = 3 + 4\mathrm{i}$	$z \cdot z^* = 25$
$z = -\sqrt{5} + 2\mathrm{i}$	$\lvert z\rvert = \sqrt{\left(-\sqrt{5}\right)^2 + 2^2} = 3$	$z = -\sqrt{5} - 2\mathrm{i}$	$z \cdot z^* = 9$
$z = a + b\mathrm{i}$	$\lvert z\rvert = \sqrt{a^2 + b^2}$	$z = a - b\mathrm{i}$	$z \cdot z^* = a^2 + b^2$

1 **Factual:** What is the relationship between the product of a complex number with its conjugate, and the modulus of the complex number?

Answer: The product is the square of the modulus.

2 **Conceptual:** Why do a complex number and its conjugate have equal moduli?

Answer: The moduli are equal since they are equally distant to the origin.

3 **Conceptual:** Verify that z and z^* are mutually conjugate.

Answer: Let $z = x + \mathrm{i}y$

Then $z^* = x - \mathrm{i}y$

and $(z^*)^* = x - (-\mathrm{i}y) = x + \mathrm{i}y = z$

(This leads to the conceptual understanding): The complex conjugate of a complex number displays equal real parts and an imaginary part equal in magnitude but opposite in sign.

TOK

Could we ever reach a point where everything important in a mathematical sense is known?

Answer: Reflect on the creation of complex numbers before their applications were known. Faith is being sure of something, often without material proof. So what is the square root of negative 1? Where does imagination come into the creation of *i*? How are intuition and reasoning involved in square rooting a number that was imaginary in the first place?

Investigation 9

Conceptual understanding:

The powers of imaginary numbers and complex numbers may be generalized to a repeating linear sequence with period four.

n	0	1	2	3	4	5	6	7	8	9
i^n	1	i	−1	−i	1	i	−1	−i	1	i

1 The values 1, i, −1, −i repeat with period 4.

2 $\mathrm{i}^n = \begin{cases} 1, n = 4k \\ \mathrm{i}, n = 4k+1 \\ -1, n = 4k+2 \\ -\mathrm{i}, n = 4k+3 \end{cases}, k \in \mathbb{N}$

3 $\mathrm{i}^{2019} = \mathrm{i}^{2016+3} = \mathrm{i}^3 = -\mathrm{i}$

4 Yes, the rule found in question 2 works for all integers. Students should try some negative integers to test; for example $-1 = 4 \cdot (-1) + 3 \Rightarrow \frac{1}{\mathrm{i}} = \mathrm{i}^{-1} = -\mathrm{i}$.

5 **Conceptual:** Can you generalize the powers of imaginary numbers?

Answer (this is the conceptual understanding): The powers of imaginary numbers and complex numbers may be generalized to a repeating linear sequence with period four.

6 $(a+b\mathrm{i})^2 = a^2 + 2ab\mathrm{i} + b^2 \underbrace{\mathrm{i}^2}_{-1} = (a^2 - b^2) + 2ab\mathrm{i}$

$(a+b\mathrm{i})^3 = a^3 + 3a^2b\mathrm{i} + 3ab^2 \underbrace{\mathrm{i}^2}_{-1} + b^3 \underbrace{i^3}_{-\mathrm{i}} = (a^3 - 3ab^2) + (3a^2b - b^3)\mathrm{i}$

7 $z^4 = (a+b\mathrm{i})^4 = a^4 + 4a^3b\mathrm{i} + 6a^2b^2 \underbrace{\mathrm{i}^2}_{-1} + 4ab^3 \underbrace{i^3}_{-\mathrm{i}} + b^4 \underbrace{i^4}_{1} = (a^4 - 6a^2b^2 + b^4) + (4a^3b - 4ab^3)\mathrm{i}$

$(z^4)^* = (a^4 - 6a^2b^2 + b^4) - (4a^3b - 4ab^3)\mathrm{i}$

$(z^*)^4 = (a-b\mathrm{i})^4 = a^4 - 4a^3b\mathrm{i} + 6a^2b^2 \underbrace{\mathrm{i}^2}_{-1} - 4ab^3 \underbrace{i^3}_{-\mathrm{i}} + b^4 \underbrace{i^4}_{1} = (a^4 - 6a^2b^2 + b^4) + (-4a^3b + 4ab^3)\mathrm{i}$

$= (a^4 - 6a^2b^2 + b^4) - (4a^3b - 4ab^3)\mathrm{i}$

Notice that $(z^4)^* = (z^*)^4$.

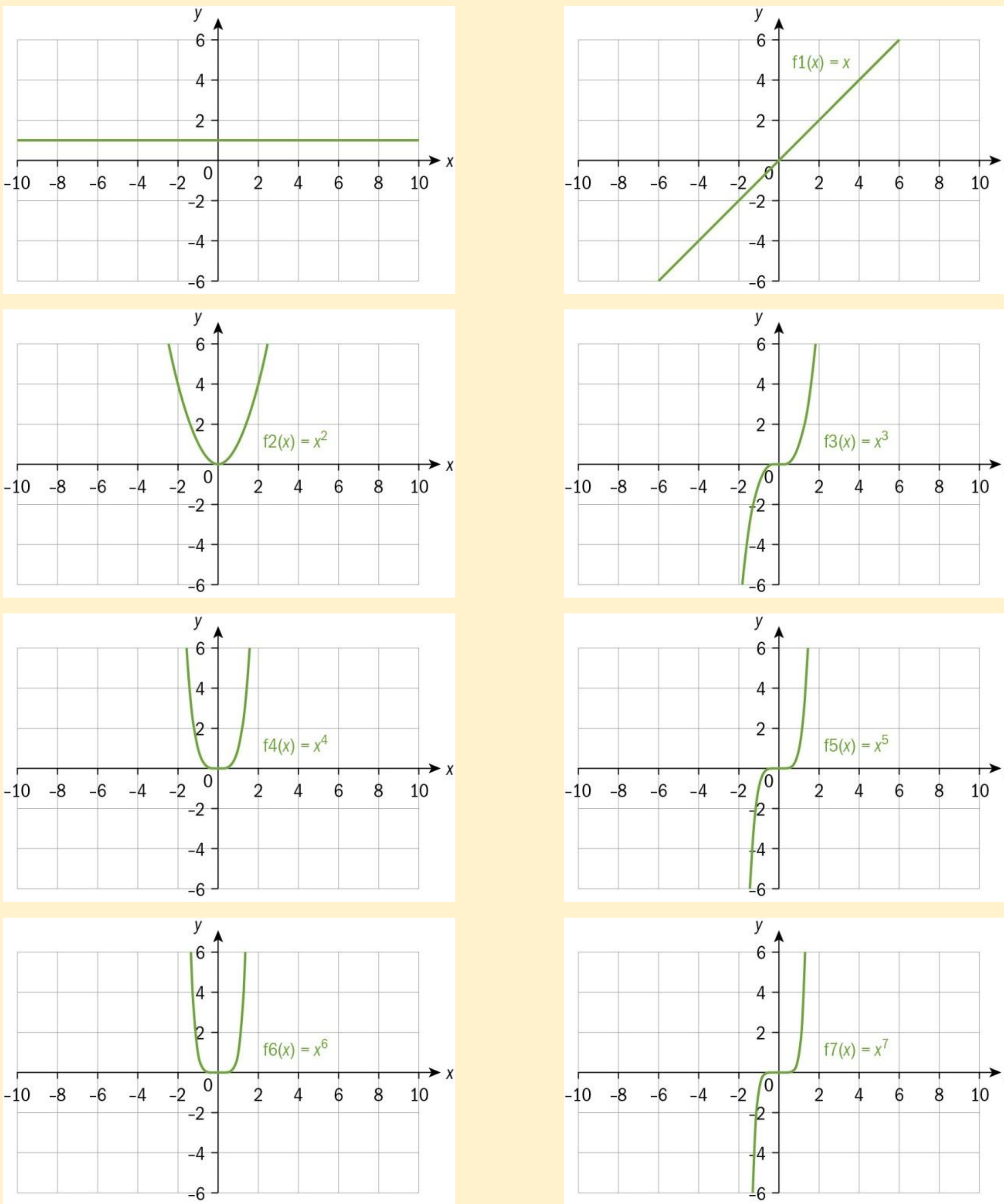
3.3 Polynomials and their graphs, Polynomials equations and inequalities

Investigation 10

Conceptual understanding:

All the even power graphs (including $n = 0$) are symmetrical about the y-axis (these are even functions) and all the odd power graphs (including $n = 1$) have rotational symmetry, order 2, about the origin (these are odd functions).

1



2 **Factual:** How could you classify these graphs by their shapes?

Answer: The first two powers n = 0,1 are lines (horizontal and oblique). The rest of the graphs can be classified into two categories: even powers have a "U-shape"; and odd powers have a "Flex shape".

3 **Conceptual:** How could you classify these graphs by their symmetrical properties?

Answer (this is the conceptual understanding): All the even power graphs (including n = 0) are symmetrical about the y-axis (these are even functions) and all the odd power graphs (including n = 1) have rotational symmetry, order 2, about the origin (these are odd functions).

Investigation 11

Conceptual understanding:

The parameters of a cubic function alter the shape, turning points and intercepts of the graph of the function.

1 Increasing a means the graph grows (for positive x) or decreases (for negative x) more quickly.

2 Changing d translates the graph in the y-direction.

3 When b = 0 the curve has a "Flex" shape at the origin.

When $b \neq 0$, the curve intersects the x-axis at two points, namely the origin (0,0) and the point (–b,0). This causes two "U" shapes with different concavity. When b > 0 the first "U" shape has a minimum at the origin, whilst when b < 0 the second "U" shape has a maximum at the origin.

4 When c = 0 the curve has a "Flex" shape at the origin.

When c > 0 the curve has a stretched "Flex" shape at the origin. When c < 0 the curves intersect the x-axis at three points; namely $\left(-\sqrt{-c},0\right)$, (0,0) and $\left(\sqrt{-c},0\right)$, making two "U" shapes with different concavity.

5 **Conceptual:** How do the parameters in quadratic and cubic functions affect their graphs?

Answer (this is the conceptual understanding): The parameters of a cubic function alter the shape, turning points and intercepts of the graph of the function.

International-mindedness

Sulba Sutras in ancient India and the Bakhshali Manuscript contained an algebraic formula for solving quadratic equations.

Reflect: What do you notice about the quotient polynomial $q(x)$ and the result you obtained from synthetic division in Example 22? Can you use the remainder theorem to explain why this is the case?

Answer: When we use synthetic division, in the last row apart from getting the remainder we also obtain the coefficients of the quotient polynomial $q(x) = 3x^2 + 5x - 17$.

The remainder theorem tells us that when $f(x)$ is divided by $x + 2$, the remainder is $f(2)$. Thus, synthetic division is effectively finding the quotient and remainder when a polynomial $f(x)$ is divided by a linear polynomial $g(x)$.

Developing Inquiry skills

Return to the opening problem. Can you use a polynomial to model the roller coaster?

Answer: The following Geogebra web applet is a fantastic tool.

https://www.geogebra.org/m/V4uXcxxn

Students can adjust the zeros of a polynomial function in order to try to create an appropriate model for a roller coaster. The program automatically calculates the equation of the polynomial function used.

Before using this, you might want to ask students what order of polynomial might best be used to model this roller coaster, and what the sign of the leading coefficient might be.

3.4 The fundamental theorem of algebra, Sum and product of the zeroes of polynomials

Investigation 12

Conceptual understanding:

Any number can be expressed as the product of prime factors. Any polynomial may be expressed as the product of one or more simple linear polynomial factors called prime factors.

1 $30 = 2\cdot 3\cdot 5$, $504 = 2^3\cdot 3^2\cdot 7$, $1155 = 3\cdot 5\cdot 7\cdot 11$, $35200 = 2^5\cdot 5^2\cdot 11$

2 $(x+5)^2$, $x(x+3)(x-5)$, $(x-1)^3$, $(x^2-1)(x^2-4) = (x-1)(x+1)(x-2)(x+2)$

3 **Conceptual:** What are the similarities and differences between factorizing a number into prime factors, and factorizing a polynomial expression?

Answer (this is the conceptual understanding): Any number can be expressed as the product of prime factors. Any polynomial may be expressed as the product of one or more simple linear polynomial factors called prime factors.

Investigation 13

Conceptual understanding:

Complex zeros of polynomial functions occur in conjugate pairs.

1 **e.g.** in part **a**, $f(\mathrm{i}) = 2\mathrm{i}^3 - 3\mathrm{i}^2 + 2\mathrm{i} - 3 = -2\mathrm{i} + 3 + 2\mathrm{i} - 3 = 0$

In a similar way, all the remainders of $f(z)$ are equal to 0 and are hence zeros of the polynomial *f*.

2 **e.g.** in part **a**, $f(-\mathrm{i}) = 2(-\mathrm{i})^3 - 3(-\mathrm{i})^2 + 2(-\mathrm{i}) - 3 = 2\mathrm{i} + 3 - 2\mathrm{i} - 3 = 0$

In a similar way, all the remainders of $f(z^*)$ are equal to 0 and are hence zeros of the polynomial *f*.

3 **Conceptual:** If *z* is a zero of a polynomial *f*, what can you say about the complex conjugate z^*?

Answer (this is the conceptual understanding): Complex zeros of polynomial functions occur in conjugate pairs.

4 **Factual:** By considering the fundamental theorem of algebra and your answer to question 3, what can you say about the zeros of an odd-degree polynomial?

Answer: A polynomial of odd degree will always have at least one real zero, since it will have an odd number of zeros, and any complex zeros will occur in conjugate pairs.

TOK

Aliens might not be able to speak an Earth language but would they still describe the equation of a straight line in similar terms?

Is mathematics a formal language?

Answer: Some questions for the development of a discussion.

What makes something a language?

Can you communicate in mathematical symbols?

Could you communicate to people, who speak a different language that you do not understand, by using mathematics?

What would Godel say? What would Hilbert say?

International-mindedness

François Viète (1540–1603) was a French amateur mathematician and astronomer, whilst **Albert Girard** was a French mathematician and musician.

Starter point for discussion: Do mathematicians have other occupations?

Investigation 14

Conceptual understanding:

The sum of the zeros of a cubic polynomial equals the opposite ratio of the quadratic coefficient and the cubic coefficient, and the product of the roots equals the opposite ratio of the constant term and the cubic coefficient.

1 $f_1 : \{-2,-1,2\}, f_2 : \{1,2,3\}, f_3 : \{-2,1,5\}, f_4 : \{-21,-11,5\}$

2 $f_1 : \sum_{k=1}^{3} x_k = -1, \prod_{k=1}^{3} x_k = 4, \quad f_2 : \sum_{k=1}^{3} x_k = 6, \prod_{k=1}^{3} x_k = 6,$

$f_3 : \sum_{k=1}^{3} x_k = 4, \prod_{k=1}^{3} x_k = -10, \quad f_4 : \sum_{k=1}^{3} x_k = -27, \prod_{k=1}^{3} x_k = 1155$

3 **Factual:** How do the sums and products of the zeros you found in question 2 relate to the coefficients of each polynomial?

Answer: The sum of zeros equals the negative value of the x^2 coefficient, i.e. $\sum_{k=1}^{3} x_k = -b$

The product of zeros equals the negative value of the constant term. $\prod_{k=1}^{3} x_k = -d$.

4 $f_5 : \left\{\frac{1}{2},2,3\right\}, f_6 : \left\{-\frac{2}{3},1,5\right\}, f_7 : \left\{-1,-\frac{2}{3},\frac{2}{5}\right\}, f_8 : \left\{-11,-\frac{21}{2},\frac{5}{7}\right\}$

5 $f_1 : \sum_{k=1}^{3} x_k = \frac{11}{2}, \prod_{k=1}^{3} x_k = 3, \quad f_2 : \sum_{k=1}^{3} x_k = \frac{16}{3}, \prod_{k=1}^{3} x_k = -\frac{10}{3},$

$f_3 : \sum_{k=1}^{3} x_k = -\frac{19}{15}, \prod_{k=1}^{3} x_k = \frac{4}{15}, \quad f_4 : \sum_{k=1}^{3} x_k = -\frac{291}{14}, \prod_{k=1}^{3} x_k = \frac{1155}{14}$

6 **Conceptual:** How do the sum and product of the roots of a cubic relate to their coefficients?

Answer (this is the conceptual understanding): The sum of the zeros of a cubic polynomial equals the opposite ratio of the quadratic coefficient and the cubic coefficient, and the product of the roots equals the opposite ratio of the constant term and the cubic coefficient.

That is, for $f(x) = ax^3 + bx^2 + cx + d$, then $\sum_{k=1}^{3} x_k = -\frac{b}{a}, \prod_{k=1}^{3} x_k = -\frac{d}{a}$.

Investigation 15

Conceptual understanding:

The sum of the zeros of a fourth-degree polynomial equals the opposite ratio of the cubic coefficient and the quartic coefficient, and the product of the roots equals the ratio of the constant term and the quartic coefficient.

1 $f_1 : \{-3, -1, 1, 2\}, f_2 : \{-7, -4, 2, 5\}, f_3 : \left\{-2, -\frac{2}{3}, \frac{1}{2}, 1\right\}, f_4 : \left\{-11, -1, \frac{2}{5}, \frac{5}{7}\right\},$

$f_5 : \left\{\frac{1}{2}, \frac{1}{2}, -4\text{i}, 4\text{i}\right\}, f_6 : \left\{-\frac{5}{2}, \frac{1}{3}, 2 - 5\text{i}, 2 + 5\text{i}\right\}$

2 $f_1 : \sum_{k=1}^{4} x_k = -1, \prod_{k=1}^{4} x_k = 6, f_2 : \sum_{k=1}^{4} x_k = -4, \prod_{k=1}^{4} x_k = 280,$

$f_3 : \sum_{k=1}^{4} x_k = -\frac{7}{6}, \prod_{k=1}^{4} x_k = \frac{2}{3} = \frac{4}{6}, f_4 : \sum_{k=1}^{4} x_k = -\frac{381}{35}, \prod_{k=1}^{4} x_k = \frac{22}{7} = \frac{110}{35},$

$f_5 : \sum_{k=1}^{4} x_k = 1, \prod_{k=1}^{4} x_k = 4 = \frac{16}{4}, f_6 : \sum_{k=1}^{4} x_k = \frac{11}{6}, \prod_{k=1}^{4} x_k = \frac{145}{6}.$

3 **Conceptual:** How do the sum and product of the zeros of a fourth degree polynomial relate to its coefficients?

Answer (this is the conceptual understanding): The sum of the zeros of a fourth-degree polynomial equals the opposite ratio of the cubic coefficient and the quartic coefficient, and the product of the roots equals the ratio of the constant term and the quartic coefficient.

That is, $f(x) = ax^4 + bx^3 + cx^2 + dx + e \Rightarrow \sum_{k=1}^{3} x_k = -\frac{b}{a}, \prod_{k=1}^{3} x_k = \frac{e}{a}$

3.5 Solving systems of linear equations

Investigation 16

Conceptual understanding:

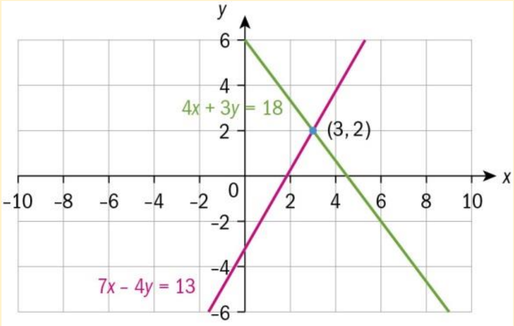
Solutions to systems of two linear equations with two unknowns can produce: a unique solution when the lines intersect at one point; no solution when the lines are parallel; and infinitely many solutions when the lines coincide.

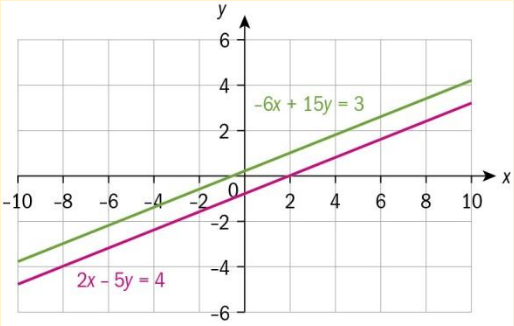
1 a $(x, y) = (3, 2)$

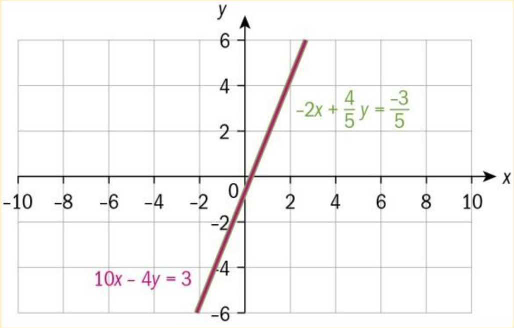
b no solution

c infinitely many solutions: any point on the line $y = \frac{5}{2}x - \frac{3}{4}$.

2







3 Factual: What are the different types of solution to a system of two linear equations with two unknowns?

Answer: Solutions to systems of two linear equations with two unknowns can produce: a unique solution; no solution; and infinitely many solutions.

4 Conceptual: Explain the geometrical significance of each type.

Answer (this is the conceptual understanding): Solutions to systems of two linear equations with two unknowns can produce: a unique solution when the lines intersect at one point; no solution when the lines are parallel; and infinitely many solutions when the lines coincide.

TOK

If we can find solutions in higher dimensions can we reason that these spaces exist beyond our sense perception?

Answer: We can consider the ways of knowing, sense perception and reason and their relation to mathematics.

We can as questions such as how many dimensions for a straight line, rectangle, cuboid? Now, how many for a dot ?

An interesting discussion at
www.researchgate.net/post/Why_cant_we_imagine_four_dimensions
and an article at
www.telegraph.co.uk/technology/3341260/Why-I-believe-in-higher-dimensions.html

TOK

How accurate is a visual representation of a mathematical concept?

Answer: A chance to explore the limitations of graphs and charts in delivering information about functions and phenomena in general, relevance of modes of representation.

A good question to ask for a response to is "Should we accept simplicity over accuracy to relay information".

Investigation 17

Conceptual understanding:

Solutions to systems of three linear equations with three unknowns can produce a unique solution, no solution and infinitely many solutions.

1 a (5, −1, 3)

b no solution

c infinitely many solution $(2y-3z+4, y, z), y, z \in \mathbb{R}$

d $\left(\frac{57}{2}, -\frac{23}{4}, -185\right)$

e infinitely many solution $\left(\frac{2z-7}{3}, \frac{4z-29}{3}, z\right), z \in \mathbb{R}$

f no solution

2 **Conceptual:** How can you classify the types of solutions to systems of three linear equations with three unknowns?

Answer (this is the conceptual understanding): Solutions to systems of three linear equations with three unknowns can produce a unique solution, no solution and infinitely many solutions.

3 **Factual:** What is the difference between the sets of infinitely many solutions to the equations in parts c and e?

Answer: In part c there is a constraint on one variable only – y and z can both vary. In part e, there is a constraint on two variables, and only z can vary.

4 **Factual:** What is the relationship between the coefficients of the three equations in part c?
Answer: Each of the three equations are linear multiples of one another.

Modelling and Investigation Activity: Making a Mandelbrot!

Approaches to Learning: Critical thinking, Communication,

Exploration Criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB Topic: Complex numbers

A lot of students choose to look at fractals in their explorations, and so Mandelbrot and Julia sets are popular (together with the Koch snowflake and the Sierpinski Triangle). The Mandelbrot set is an intriguing construction and an interesting avenue to explore but is not always well executed if students do not take the time to understand its construction and then, importantly, to think of ways they can extend their problem.

The mathematics is relatively easy to understand. This activity is designed to help students take the first steps on this intriguing discovery.

Fractals

To introduce this topic, you could ask:

Have you heard of fractals?

What do you know about them?

Exploring an iterative equation

Students will now consider iterative equations to increase their understanding of how a particular fractal is created.

$z_2 = 0.5$

$z_3 = 0.75$

$z_4 = 1.0625$

$z_5 = 1.6289...$

$z_6 = 3.15333...$

$z_7 = 10.44352...$

The values found in this calculation are getting larger and larger – towards infinity.

A different iterative equation

$z_2 = -0.5$

$z_3 = -0.25$

$z_4 = -0.4375$

$z_5 = -0.30859...$

$z_6 = -0.40476...$

$z_7 = -0.33616...$

The values found in this calculation stay within a boundary, converging.

The Mandelbrot set

You could ask:

Can you find another integer value of c that would mean that the process would zoom off to infinity?

For example, $c = 1$

Can you find another integer value c where the process wouldn't zoom off to infinity?

For example, $c = -1.3, -1.1$

$z_2 = 1 + i$

$z_3 = 1 + 3i$

$z_4 = -7 + 7i$

$z_5 = 1 - 97i$

$z_6 = -9407 - 193i$

$z_7 = 88\ 454\ 401 - 3\ 631\ 103i$

Clearly this zooms off to infinity.

a $c = 0.2 - 0.7i$

a	+	b	i
0	+	0	i
0.2	+	-0.7	i
-0.25	+	-0.98	i
-0.6979	+	-0.21	i
0.6429644	+	-0.406882	i
0.4478503	+	-1.2232213	i
-1.0957005	+	-1.79564	i

This is heading off to infinity (slowly), so is **not** in the Mandelbrot set.

b $c = -0.25 + 0.5i$

a	+	b	i
0	+	0	i
-0.25	+	0.5	i
-0.4375	+	0.25	i
-0.1210938	+	0.28125	i
-0.3144379	+	0.4318848	i
-0.3376533	+	0.2283982	i
-0.188156	+	0.3457612	i
-0.3341482	+	0.3698859	i
-0.2751606	+	0.2528066	i
-0.2381978	+	0.3608752	i
-0.3234927	+	0.3280806	i
-0.2529894	+	0.2877366	i
-0.2687887	+	0.3544114	i
-0.30336	+	0.3094764	i
-0.2537484	+	0.3122344	i
-0.2831021	+	0.3415421	i
-0.2865042	+	0.3066174	i
-0.2619296	+	0.3243057	i
-0.286567	+	0.3301095	i
-0.2768516	+	0.310803	i
-0.2699517	+	0.3279074	i
-0.2846493	+	0.3229617	i
-0.273279	+	0.3161383	i

This is not heading off to infinity, so **is** in the Mandelbrot set.

If c = i:

a	+	b	i
0	+	0	i
0	+	1	i
-1	+	1	i
0	+	-1	i
-1	+	1	i
0	+	-1	i
-1	+	1	i
0	+	-1	i
-1	+	1	i
0	+	-1	i
-1	+	1	i
0	+	-1	i

This is not going to infinity, so **is** in the Mandelbrot set.

Students could find an efficient way of trying some of their own values on a calculator, or start thinking about a spreadsheet to do the calculations. There will also hopefully be a comment regarding the fact that it is easier to spot the values heading off to infinity (or not) with some values rather than others.

With a class it can be good to have a set of axes for an Argand diagram on the board (say from -2 to 2 on the real axis and -2 to 2 on the imaginary axes). Students can then come up to the board when they have calculated a point and colour it according to whether it is in the Mandelbrot set or not – slowly (very slowly!) the set will take shape!

Here is an example of a programme on Geogebra: https://www.geogebra.org/m/vK8zhJKM.

Extensions

The Mandelbrot and the related Julia set is a popular topic for mathematics explorations. It is important therefore that students do more than just explain how it is formed.

You could ask:

What other questions could you ask?

What other approaches could you pursue to make your own exploration?

A few suggestions are given in the task in the Students Book.

Some of these extensions may take students well beyond mathematics they are comfortable with, and therefore they may find it difficult to demonstrate understanding. You may need to advise them carefully on this if their ideas are too ambitious.

4 Measuring change: differentiation

Essential understandings

Calculus describes rates of change between two variables and the accumulation of limiting areas. Understanding these rates of change and accumulations allow us to model, interpret and analyze real-world problems and situations. Calculus helps us to understand the behaviour of functions and allows us to interpret the features of their graphs.

Content specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- The derivative may be represented physically as a rate of change and geometrically as the gradient or slope function.
- Examining rates of change close to turning points helps to identify intervals where the function increases/decreases, and identify the concavity of the function.
- Mathematical modelling can provide effective solutions to real-life problems in optimization by maximizing or minimizing a quantity, such as cost or profit.
- Derivatives describe real-world kinematics problems in two and three-dimensional space by examining displacement, velocity and acceleration.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
A limit describes the output of a function as the input approaches a certain value from the left and from the right.	Investigation 2
For the limit of a function to exist, as the input of the function approaches a certain value from the left and from the right, the outputs must approach the same real number.	Investigation 2
Examining the limit of a function at a point, and the value of the function at that point, determines continuity at the point.	Investigation 3
Examining the limits of functions at infinity can help define the oblique asymptote of a function.	Investigation 4
Comparing the degree of the numerator in relationship to the degree of the denominator can help determine if the rational function has a horizontal asymptote.	Investigation 5
A sequence converges if its sequential terms approach a finite limit.	Investigation 6
An average rate of change describes the ratio between the total change in the output values and the total change in the input values.	Investigation 7

Tangents to a curve at a point identify the instantaneous rate of change of its function at specific values of the independent variable.	Investigation 8
Examining local linearity provides a visual approach to differentiability and continuity at a point.	Investigation 9
Some functions may be continuous everywhere but not differentiable everywhere.	Investigation 10
A constant function, i.e., a line parallel to the x-axis, has gradient 0.	Investigation 11
The parameter m represents the gradient of a straight line in the formula $y = mx + c$, hence m represents the derivative of the linear function.	Investigation 11
The degree of the derivative of a polynomial function differs from the degree of the function by 1.	Investigation 12
The product rule can be used to find the derivative of a rational function by writing the quotient as the product of two algebraic expressions.	Investigation 14
By investigating patterns of higher derivatives of a function, useful rules emerge for finding the *n*th derivative of a function that facilitate the analytical process.	Investigation 15
Examining the signs of the gradients to the left and right of stationary points help determine the nature of stationary points, and identify the intervals where the function is increasing/decreasing.	Investigation 16
The second derivative describes the rate of change of the gradient function (first derivative); therefore at a stationary point an increasing first derivative function represents a minimum while a decreasing first derivative function represents a maximum.	Investigation 17
When the 2nd derivative test fails to determine the nature of a stationary point, revert to testing the gradient on each side of the stationary point to test for a point of inflexion.	Investigation 17
Examining the signs of the second derivative at a stationary point of f' can help identify the intervals where the function is concave up/down since the second derivative indicates if the first derivative is increasing/decreasing.	Investigation 18
At a horizontal point of inflexion *c*, $f'(c) = f''(c) = 0$ and the concavity of f changes as *x* passes through *c*.	Investigation 19

Syllabus sections covered in this chapter:

- SL5.1*
- SL5.2*
- SL5.3*
- SL5.4*
- SL5.6
- SL5.7
- SL5.8
- SL5.9
- AHL5.12
- AHL5.14

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and the development of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 219: rational exponents, graph sketching, sums of infinite geometric series.	Page 242: Example 13 Page 254: Example 20 Page 263: Example 26 Page 280: Example 34 Page 286: Example 36 Page 290: Example 37	Page 222: Example 1 Page 231: Example 7 Page 234: Example 9 Page 263: Example 26 Page 271: Example 30 Page 280: Example 34	Pages 235, 248, 261, 277, 287, 294

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 295	Page 298	N/A

4.1 Limits, continuity and convergence

TOK

"This statement is false."

Should paradoxes change the way that mathematics is viewed as an area of knowledge?

Answer: Paradoxes might change your opinion about how mathematics is seen as an area of knowledge.

You might want students to try to write their own liar paradoxes such as "I am lying".

This could be an opening discussion to be linked to Zeno's paradox of Achilles and the tortoise in 3.1.

Investigation 1

Note: This investigation is meant to give an 'in-context' introduction to the concept of limits, and does not have a conceptual understanding.

1 a 10 m

b 1 m

c 0.1 m

2 0.01

3 0.001

4 10 + 1 + 0.1 + 0.01 + 0.001; $a = 10$, $r = 0.1$

5 $|r| < 1$

6 Yes, since $r = 0.1 < 1$; $S_\infty = \frac{a}{1-r} = \frac{10}{1-0.1} = 11.11\text{m}$

7 Yes, after 11.11 m.

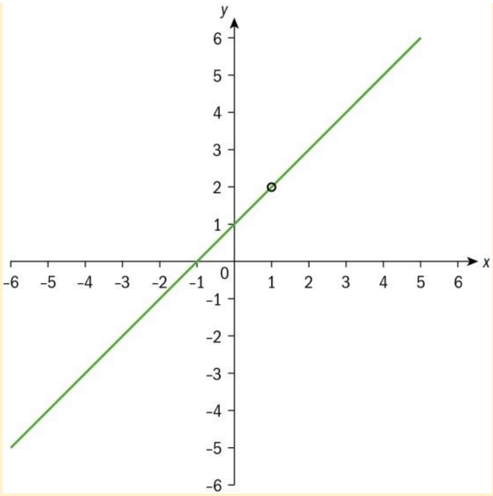
Investigation 2

Conceptual understandings:

A limit describes the output of a function as the input approaches a certain value from the left and from the right.

For the limit of a function to exist, as the input of the function approaches a certain value from the left and from the right, the outputs must approach the same real number.

1 Note that the point (1, 2) is not included in the domain of *f*.



2

x	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
$f(x)=\dfrac{x^2-1}{x-1}$	1.6	1.7	1.8	1.9	Undef.	2.1	2.2	2.3	2.4

3 2

4 **Conceptual:** What is meant by the limit of a function at a particular point?

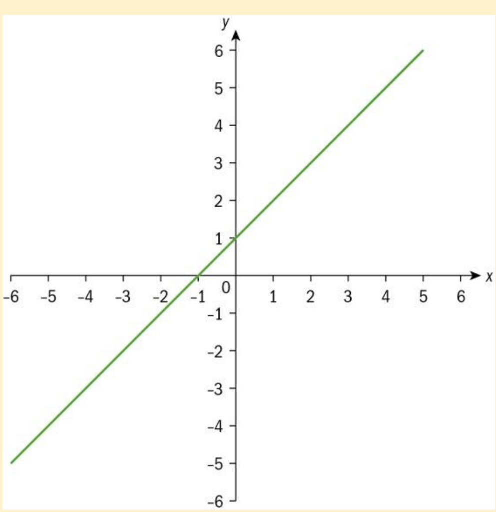
Answer (this is a conceptual understanding): A limit describes the output of a function as the input approaches a certain value from both the negative and positive directions.

5 The function approaches the same value of 2 as x approaches 1 from the negative direction and from the positive direction, although the function itself is not defined at $x = 1$.

6 **Conceptual:** What condition is placed on the left and right limits of f at $x = a$ for the limit of f to exist at a?

Answer (this is a conceptual understanding): For the limit of a function to exist, as the input of the function approaches a certain value from the negative and positive directions, the outputs must approach the same real number.

7 Note that $x = 1$ is included in the domain here.



x	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
$g(x) = x + 1$	1.6	1.7	1.8	1.9	2	2.1	2.2	2.3	2.4

8 **Conceptual:** Does a function need to be defined at $x = c$ in order to have a limit at $x = c$?

Answer: No.

9 **Factual:** Write down graphical similarities and differences between *f* and *g*, and explain. Can you see the graphical difference between *f* and *g* on your GDC or graphing software? If not, explain why not.

Answer: The graphs of *f* and *g* are identical except at $x = 1$, as *f* is undefined at this value, and *g* is defined at this value.

TOK

What value does the knowledge of limits have?

Answer: Is infinitesimal behaviour applicable to real life?

Are intuition and imagination valid ways of knowing in mathematics?

Investigation 3

Conceptual understandings:

Examining the limit of a function at a point, and the value of the function at that point, determines continuity at the point.

1 This investigation will lead students to establish a rigorous definition of continuity. At this stage, students might informally identify a continuous function as one that has no 'gaps' or 'holes' in its graph. In other words, the function is defined at every point in its domain and it's graph has no sudden 'breaks'.

2 a Both are 1

b Both are 5.

At $x = 0$ and $x = -1$, the value of $f(x)$, and the limit of $f(x)$ at that point, are equal.

3 **Conceptual:** What two things do you need to consider in order to determine whether *f* is continuous at a given point?

Answer (this is the conceptual understanding): Examining the limit of a function at a point, and the value of the function at that point, determines continuity at the point.

4 a Yes, since the domain of $f(x)$ is the set of real numbers.

b Yes, since the function is continuous; it has no breaks or jumps on its graph.

5 **Conceptual:** Using your results from **4a** and **4b** what generalization can you make about any continuous function *f*?

Answer: For any point *a* on the domain of *f*, both f(a) and $\lim_{x \to a} f(x)$ exist, and

$\lim_{x \to a} f(x) = f(a)$.

6 a 2

b undefined.

$\lim_{x \to 1} f(x) \neq f(1)$ since $f(1)$ is not defined. Therefore, this equality must be a necessary condition for continuity.

7 a The left limit is 4 and the right limit is 1, so $\lim_{x \to 2} f(x)$ does not exist.

b $f(2)$ is undefined as 2 is not in the domain of f.

8 **Conceptual:** Consider all your answers to the questions and write down three conditions necessary for a function to be continuous at $x = a$.

Answer: $\lim_{x \to a} f(x)$ exists; $f(a)$ is defined; and $\lim_{x \to a} f(x) = f(a)$.

Reflect: Of the types of functions you studied in Chapter 2 – polynomial functions, rational functions, modulus functions, and radical functions – which are continuous functions over the set of real numbers, and which functions have discontinuities at certain points?

Answer: Polynomial functions and modulus polynomial functions are everywhere continuous.

Rational functions are continuous on their domain, but not everywhere continuous as they have points of discontinuity.

Radical functions ($f(x) = \sqrt[n]{x}$) are continuous on their domain, but not everywhere continuous since they are not defined for negative x.

International-mindedness

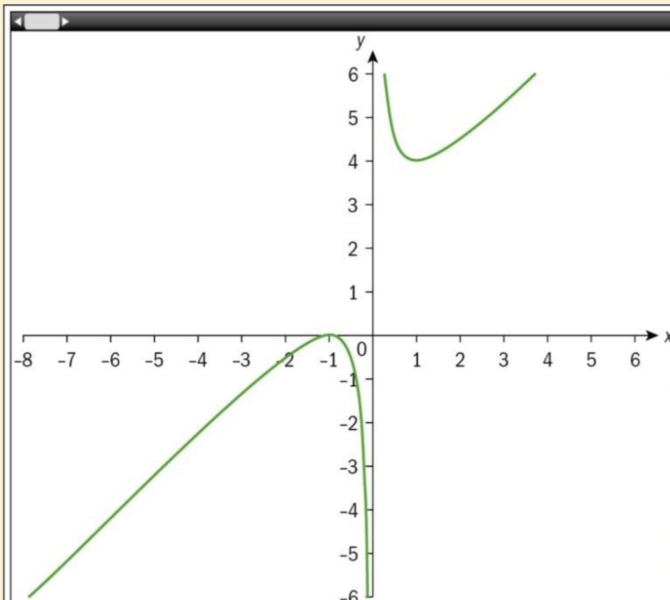
Maria Agnessi, an 18th century, Italian mathematician, published a text on calculus and also studied curves of the form $y = \dfrac{a^2}{x^2 + a^2}$.

Investigation 4

Conceptual understanding:

Examining the limits of functions at infinity can help define the oblique asymptote of a function.

1



x	$f1(x) = \frac{x^2+2x+1}{x}$
1	4
2	4.5
3	5.33333
4	6.25
5	7.2
6	8.16667
7	9.14286
8	10.125
9	11.1111
10	12.1
11	13.0909
12	14.0833
13	15.0769
14	16.0714
15	17.0667
16	18.0625
17	19.0588
18	20.0556
19	21.0526
20	22.05
21	23.0476

As x approaches $\pm\infty$, the y-value is approximately 2 more than the x-value.

2 $y = x + 2$

3 $f(x) = \frac{x^2 + 2x + 1}{x} = x + 2 + \frac{1}{x}$; $\lim_{x \to \infty}\left(x + 2 + \frac{1}{x}\right) = \lim_{x \to \infty}(x + 2)$; limit of f as x approaches $\pm\infty$ is the line $y = x + 2$.

4 **Conceptual:** What special property of a graph can you find by examining the limit of the function as x tends to infinity?

Answer (this is the conceptual understanding): Examining the limits of functions at infinity can help define the oblique asymptote of a function.

Investigation 5

Conceptual understanding:

Comparing the degree of the numerator in relationship to the degree of the denominator can help determine if the rational function has a horizontal asymptote.

1 a $y = -2$

b $y = 0$

c None

d $y = 0$

e $y = \frac{1}{4}$

f None

g $y = 0$

h None

2

	Degree ($f(x)$)	Leading coefficient	Degree ($g(x)$)	Leading coefficient	Horizontal asymptote
a	1	−2	1	1	$y = -2$
b	1	0	2	−3	$y = 0$
c	3	1	2	1	none
d	1	0	2	1	$y = 0$
e	3	1	3	4	$y = \frac{1}{4}$
f	2	1	1	1	none
g	2	1	5	1	$y = 0$
h	5	1	3	1	none

3 **Conceptual:** Is there a rule for finding the horizontal asymptotes for some types of rational functions?

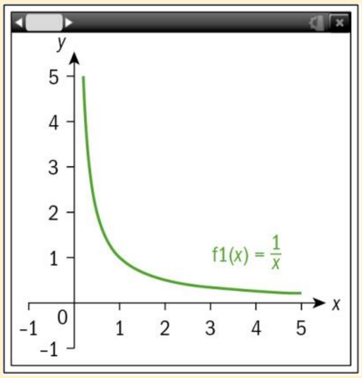
Answer (this is the conceptual understanding): Comparing the degree of the numerator in relationship to the degree of the denominator can help determine if the rational function has a horizontal asymptote.

Investigation 6

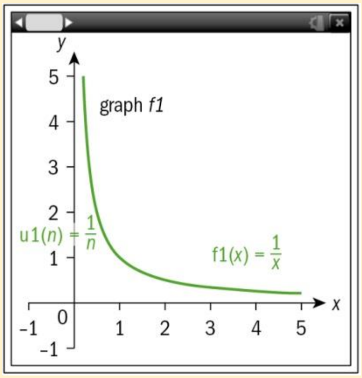
Conceptual understanding:

A sequence converges if its sequential terms approach a finite limit.

1 The limit is 0.



2



3 The limit is 0.

4 $1, \frac{1}{2}, \frac{1}{3}, \ldots; \frac{1}{m} > \frac{1}{n}$

5 Since $\frac{1}{m} > \frac{1}{n}$, the terms of the sequence are getting smaller, approaching 0.

6 **Conceptual:** When does a sequence converge?

Answer (this is the conceptual understanding): A sequence converges if its sequential terms approach a finite limit.

Developing inquiry skills box

Refer back to opening problem: How does the concept of limits help you analyze the problem of finding a runner's speed at a specific instant in time?

Answer: The limit allows you to zoom closer and closer into smaller and smaller time increments in order to find speed at any particular instant in time.

4.2 The derivative of a function

Investigation 7

Conceptual understanding:

An average rate of change describes the ratio between the total change in the output values and the total change in the input values.

1 In 2009, Bolt was slower to cover the first 30 m than in 2008, but in 2009 he completed the final 70 m quicker than he did in 2008.

2 2008: 10.3 m s^{-1}; 2009: 10.4 m s^{-1}

3 2009; 12.35 m s^{-1} between 60 m and 70 m

4 **Factual:** Compare Bolt's fastest speed with his average speed in that race. If you were to examine Bolt's speed at any point in that race, which of the two speeds you found would be likely be closest to the speed that he was running at that instant?

Answer: Average speed; this gives a good indication of the speed he was running at over the whole race. He only achieved the highest speed at one point in the race.

5 **Factual:** From the given information, is it possible to find his fastest speed at any particular moment in time, e.g., at 3.25s?

Answer: No, the best we can do is find his average speed over each 10 m interval. This is not the same as his instantaneous speed.

6 **Conceptual:** What is an average rate of change between output and input values of a function?

Answer (this is the conceptual understanding): An average rate of change describes the ratio between the total change in the output values and the total change in the input values.

Investigation 8

Conceptual understanding:

Tangents to a curve at a point identify the instantaneous rate of change of its function at specific values of the independent variable.

1

x	$B(x, f(x))$	Gradient of $[AB]$
2	(2,4)	$\frac{4-1}{2-1}=3$
1.5	(1.5, 2.25)	2.5
1.1	(1.1, 1.21)	2.1
1.01	(1.01,1.0201)	2.01
1.001	(1.001,1.002)	2.001

2 The gradient of $[AB]$ approaches 2

3

x	$B(x, f(x))$	Gradient of $[AB]$
0	(0,0)	1
0.8	(0.8, 2.25)	1.8
0.9	(0.9, 1.21)	1.9
0.999	(0.999,1.0201)	1.999

From the left, the gradient of [AB] also approaches 2.

4 $\dfrac{(x+h)^2 = x^2}{(x-(x+h))} = 2x + h$

5 **Conceptual:** How is the gradient of a tangent to the curve at a point related to the gradient of a secant line passing through the same point?

Answer: The gradient of the secant line between two points approaches the gradient of the tangent at one of these two points as the difference in the x-coordinates between the two points approaches 0.

6 $\lim\limits_{h\to 0}(2x+h) = 2x$. Hence the gradient of the tangent to $y = x^2$ at any point x is $y = 2x$

7 2

8 **Conceptual:** What does the gradient of the tangent to the curve $y = f(x)$ at any point tell you about the instantaneous rate of change between x and y at that point?

Answer (this is the conceptual understanding): Tangents to a curve at a point identify the instantaneous rate of change of its function at specific values of the independent variable.

TOK

Who do you think should be considered the discoverer of calculus?

Answer: The debate over whether Newton or Leibnitz discovered certain calculus concepts.

An opportunity to have students research and present a document/wall display.

Instructions to students might include:

- I think it was Newton or Leibniz
- You find one more person from a calculus timeline.
- Take on the role of their representative and in one paragraph, state their case with reasons and evidence.
- Now take on the role of the judge and in one paragraph, stet who you think should be called the discoverer of calculus and why.

You might want a picture of the mathematicians with their paragraphs and a picture of the student with their summary. This will see students taking ownership of their decisions that might even be called "risk taking".

Reflect: If you have two real-life physical quantities x and y which change in relation to one another, what does the derivative function tell you about these quantities?

Answer: The derivative may be represented physically as a rate of change between x and y.

Reflect: If you plot a graph of x against y, what geometrical property of the graph would the derivative function tell you?

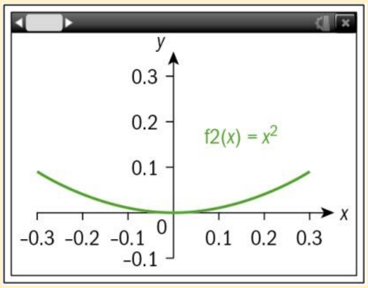
Answer: The derivative may be represented geometrically as the gradient or slope function.

Investigation 9

Conceptual understanding:

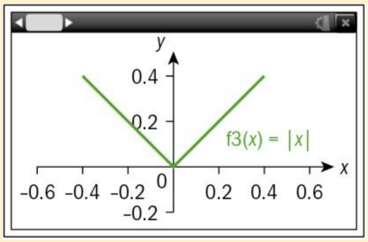
Examining local linearity provides a visual approach to differentiability and continuity at a point.

1



The curve 'flattens' at whichever point you choose; becoming linear the more you zoom in.

2



The graph remains the same 'V-shape' no matter how close you zoom in.

3 **Conceptual:** How does zooming in at a point on the graph of a function help you determine continuity and differentiability of the function at a given point?

Answer (this is the conceptual understanding): Examining local linearity provides a visual approach to differentiability and continuity at a point.

TOK

Mathematics and the real world: the seemingly abstract concept of calculus allows us to create mathematical models that permit human feats, such as getting a man on the Moon. What does this tell us about the links between mathematical models and physical reality?

Answer: To explore the relation between mathematical models and reality, the essential issue of mathematical modelling is dependent on social and personal construction processes where absolute agreement cannot be expected.

A prime pragmatic function of such models is to enable calculations and predictions of physical phenomena. Mathematical models are useful also in representing aspects of reality that are hard to visualize. Models serve as conceptual frameworks that can lead to important physical discoveries.

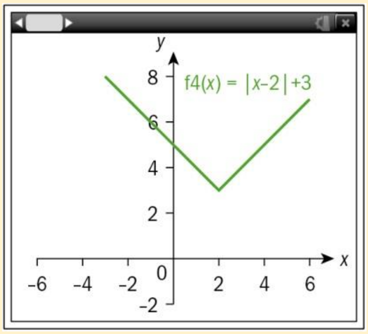
A counterclaim would be that a variety of different mathematical models can account for the same appearances, for example, the models of Ptolemy and Copernicus. This implies that the construction of models involves a substantial element of creative imagination.

Investigation 10

Conceptual understanding:

Some functions may be continuous everywhere but not differentiable everywhere.

1 a



(2,3);

There are different left-hand and right-hand limits of the derivative function:

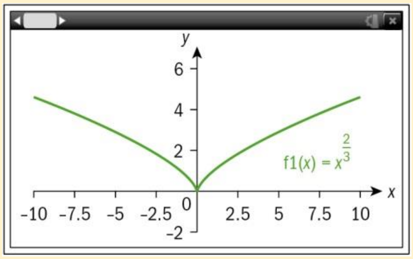
Right-hand limit: $\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{(2+h-2)-(2-2)}{h}$

$= \lim_{h\to 0^+} \frac{h}{h} = 1$

Left-hand limit: $\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{-(2+h-2)+(2-2)}{h}$

$= \lim_{h\to 0^-} \frac{-h}{h} = -1$

b



(0,0);

The value of the derivative function approaches +∞ from one side and -∞ from the other.

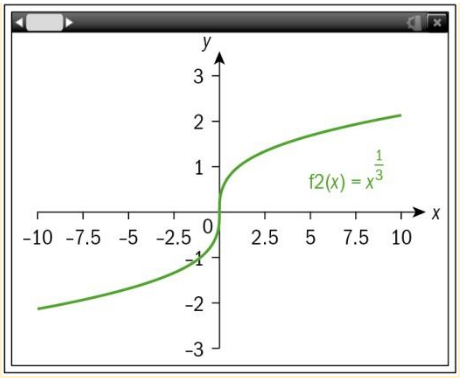
Right-hand limit: $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(0+h)^{\frac{2}{3}}-0^{\frac{2}{3}}}{h}$

$= \lim_{h\to 0^+} \frac{1}{\sqrt[3]{h}} \to +\infty$ as $h \to 0$

Left-hand limit: $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(0+h)^{\frac{2}{3}}-0^{\frac{2}{3}}}{h}$

$= \lim_{h\to 0^-} \frac{1}{\sqrt[3]{h}} \to -\infty$ as $h \to 0$

c $f(x) = \sqrt[3]{x}$



(0,0); The value of the derivative function approaches +∞ from both sides.

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{(0+h)^{\frac{1}{3}}-0^{\frac{1}{3}}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h^{\frac{2}{3}}}\to +\infty \text{ as } h\to 0 \text{ from either side}$$

2 **Conceptual:** Are all functions that are continuous throughout their domain also differentiable throughout their domain?

Answer (this is the conceptual understanding): Some functions may be continuous everywhere but not differentiable everywhere.

3 **Conceptual:** What generalizations could you make about the nature of points where functions are continuous but not differentiable?

Answer: Functions can be continuous but not differentiable when the left- and right-hand limits of the function itself are equal, but the left- and right-hand limits of the derivative function are not equal , i.e., the limit of the derivative function approaches +∞ from one side and -∞ from the other side

The gradient of the secant lines approach either +∞ or -∞.

4 **Factual:** Identify which of the three functions in question 1 has which type of point. Describe what happens to the derivative function at each of these three points.

Answer: A corner is where the left and right limits are different. A cusp is where the gradients of the secant lines approach +∞ from one side and -∞ from the other. A vertical tangent is where the gradients of the secant lines approach either +∞ or -∞.

a has a corner, **b** has a cusp, and **c** has a vertical tangent.

Reflect: Can you find other examples of how tangent and normal lines relate to real-life problems?

Answer: There are very many examples in maths and the sciences, and students could try to think of some examples in groups, or research them online.

Developing inquiry skills

Refer back to the opening problem in which a sprinter was running a race.

How does the concept of limits help you analyse the problem of finding a runner's speed at a specific instant in time?

Answer: Start by plotting a graph of the sprinter's distance against time over the course of the race. This will not be a straight line, since the speed of the sprinter varies.

To find their speed at time *t,* examine their average speed over a small time interval between t and $t + h$. This is given by $v(t) = \frac{s(t+h)-s(t)}{h}$.

Then their instantaneous speed at t is given by $v(t) = \lim_{t \to 0} \frac{s(t+h)-s(t)}{h}$.

4.3 Differentiation rules

Investigation 11

Conceptual understanding:

A constant function, i.e., a line parallel to the x-axis, has gradient 0.

The parameter *m* represents the gradient of a straight line in the formula *y*=m*x*+c, hence m represents the derivative of the linear function.

1 **Factual:** What is the *gradient* of a straight line parallel to the *x*-axis?

Answer: zero

2 **a-d** The derivative of each function is zero.

3 **Conceptual:** What is the derivative of a constant function?

Answer (this is a conceptual understanding): A constant function, i.e., a line parallel to the *x*-axis, has *derivative* 0.

4 **Factual:** For a straight line with equation $y = mx + c$, which parameter tells you the gradient of the line?

Answer: *m*

5 a −1

b 2

c $-\frac{1}{2}$

6 **Conceptual:** How is the derivative of a linear function related to the gradient of its graph?

Answer (this is a conceptual understanding): The parameter *m* represents the gradient of a straight line in the formula $y = mx + c$, hence *m* represents the derivative of the linear function at every point.

TOK

Mathematics – invented or discovered?

If mathematics is created by people, why do we sometimes feel that mathematical truths are objective facts about the world rather than something constructed by human beings?

Answer: Ask students to write down three things that were invented and three things that were discovered.

Inventions might include the airplane, lightbulbs, computer and discoveries might include dinosaur fossils, magnets, the source of the Nile.

Ask students to create and share their own definitions.

We are now approaching one of the big TOK questions: is mathematics discovered or invented?

An invention is something that was not previously there. A discovery concerns something that already exists at the time of discovery but was previously unknown. As a result of the discovery, nothing has changed apart from an associated increase in knowledge.

Is there something that is in the intersection? What about music?

Reflect: Suppose $f(x) = u(x) - v(x)$. Can you prove that $f'(x) = u'(x) - v'(x)$?

Answer: Use differentiation from first principles to form a rigorous proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{u(x+h)+v(x+h)-u(x)-u(y)}{h} \\ &= \lim_{h\to 0}\frac{u(x+h)-u(x)+v(x+h)-u(y)}{h} \\ &= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}+\lim_{h\to 0}\frac{v(x+h)-v(x)}{h} \quad \text{by properties of limits} \\ &= u'(x)+v'(x) \end{aligned}$$

Reflect: Suppose f is a linear multiple of a differentiable function g, for example $f(x)=(cg)(x)$ where c is a constant. Can you prove that $f'(x)=c(g'(x))$?

Answer: Students should start with the definition $f'(x)=\lim_{h\to 0}\frac{(cg)(x+h)-(cg)(x)}{h}$ and use properties of functions to give the required result.

Investigation 12

Conceptual understanding:

The degree of the derivative of a polynomial function differs from the degree of the function by 1.

1 a $y = 2x$

b

x	−2	−1	0	1	2	3
$\frac{dy}{dx}$	−4	−2	0	2	4	6

2 a $y = 3x^2$

b

x	−2	−1	0	1	2	3
$\frac{dy}{dx}$	12	3	0	3	12	27

3 $y = 4x^3$

x	−2	−1	0	1	2	3
$\frac{dy}{dx}$	-32	-4	0	4	32	108

4 From questions **1**, **2** and **3**, students may be able to conjecture that $y' = nx^{n-1}$. In questions **5, 6** and **7**, we use the Binomial Theorem to demonstrate a rigorous way of showing this.

5 **Conceptual:** What is the relationship between the degree of a polynomial function and the degree of its derivative?

Answer: The derivative of a polynomial function has degree 1 less than the polynomial.

6 $\frac{dy}{dx} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$

7 $\frac{dy}{dx} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$

$$= \lim_{h\to 0}\frac{\left(\left[(x+h)-x\right]\right)\left((x+h)^{n-1}+(x+h)^{n-2}x+(x+h)^{n-3}x^2+\ldots+(x+h)x^{n-2}+x^{n-1}\right)}{h}$$

$$= \lim_{h\to 0}\left((x+h)^{n-1}+(x+h)^{n-2}x+(x+h)^{n-3}x^2+\ldots+(x+h)x^{n-2}+x^{n-1}\right)$$

$$= \left((x)^{n-1}+(x)^{n-2}x+(x)^{n-3}x^2+\ldots+(x)x^{n-2}+x^{n-1}\right)$$

$$= nx^{n-1}$$

8 **Conceptual:** How is the power rule useful in differentiating a polynomial function?

Answer (this is the conceptual understanding): The power rule provides an efficient method for finding the derivative of a polynomial function without going back to first principles.

Reflect: What is the result when $n = 0$?

Answer: When $f(x) = x^0$ then $f'(x) = 0x^{-1} = 0$

Investigation 13

This investigation is to guide the student in discovering the chain rule, and to see its usefulness in avoiding tedious algebraic manipulation.

1 $\frac{dy}{dx} = 2(1+x)$

2 a $\frac{dy}{dx} = -4(1-2x)$

b $\frac{dy}{dx} = 6(3x-1)$

c $\frac{dy}{dx} = 2a(1+ax)$

3 a $y = f(g(x))$ where $g(x) = (1-2x); f(x) = x^2$

b $y = f(g(x))$ where $g(x) = (3x-1); f(x) = x^2$

c $y = f(g(x))$ where $g(x) = (1+ax); g(x) = x^2$

4 Students may spot that if $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)).g'(x)$. If so, they can test their expression using questions **5** and **6**. However, if this is too big a step for them, they can progress immediately onto question **7** which will guide them through finding an expression for the derivative of a composite function.

5 $\dfrac{dy}{dx} = -8(1-2x)^3$

6 $\dfrac{dy}{dx} = -30(1-2x)^{14}$

7 a $u = 3x - 1; y = u^2$

b $\dfrac{dy}{du} = 2u$

c $\dfrac{du}{dx} = 3$

d $$\begin{aligned}\frac{dy}{du}\cdot\frac{du}{dx} &= 2u\cdot 3\\ &= 2(3x-1)\times 3\\ &= 6(3x-1)\\ &= \frac{dy}{dx}\end{aligned}$$

Investigation 14

Conceptual understanding:

The product rule can be used to find the derivative of a rational function by writing the quotient as the product of two algebraic expressions.

1 $\dfrac{dy}{dx} = 3x^2$

2 No

3 Students' own examples.

4 **Conceptual:** What can you conclude about the derivative of a product of functions?

Answer: In general, the derivative of a product of functions is not the same as the product of the derivative of the functions.

5 a $f'(x) = \lim\limits_{h\to 0}\dfrac{u(x+h)v(x+h) - u(x)v(x)}{h}$.

b $f'(x) = \lim\limits_{h\to 0}\dfrac{u(x+h)v(x+h) + \lfloor u(x+h)v(x) - u(x+h)(v(x)\rfloor - u(x)v(x)}{h}$. Since you are adding and subtracting the same term, it does not change the value of the expression.

c $f'(x) = \lim\limits_{h\to 0}\left\lfloor u(x+h)\dfrac{v(x+h)-v(x)}{h} + v(x)\dfrac{u(x+h)-u(x)}{h}\right\rfloor$

d Letting $x \to 0$ then

$$f'(x) = u(x)v'(x) + v(x)u'(x)$$

6 **Conceptual:** How can the product rule help you find the derivative of a product of functions?

Answer (this is the conceptual understanding): The product rule can be used to find the derivative of a function by writing the quotient as the product of two algebraic expressions.

Reflect: What would the product rule be if you had three differentiable functions in x?

Answer: Suppose $f(x) = u(x)v(x)w(x)$

Then $f'(x) = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x)$

You could encourage students to prove this using differentiation from first principles.

TOK

What is the difference between inductive and deductive reasoning?

Answer: Deductive reasoning usually follows steps. A suitable definition is that deductive reasoning is the process by which a person makes conclusions based on previously known facts.

You might want to give examples such as All dolphins are mammals, all mammals have kidneys; therefore, all dolphins have kidneys, or $x = y$ and $y = z$, therefore, $x = z$.

Inductive reasoning is the opposite of deductive reasoning. Inductive reasoning draws conclusions based on a set of observations. This might not be a valid method of proof. Just because you observe a number of situations in which a pattern exists doesn't mean that that pattern is true for all situations.

Examples such as "60% of the students in your class like strawberries, so 60% of people in the world like strawberries", or "Ali's mother and father are doctors, Ali is a doctor, so Ali's brother must be a doctor".

Investigation 15

Conceptual understanding:

By investigating patterns of higher derivatives of a function, useful rules emerge for finding the nth derivative of a function that facilitate the analytical process.

1 a $f'(x) = 4x^3$

$f''(x) = 12x^2$

$f'''(x) = 24x$

$f^{(4)}(x) = 24$

b 0

c $f'(x) = kx^{k-1}$

$f''(x) = k(k-1)x^{k-2}$

$f'''(x) = k(k-1)(k-2)x^{k-3}$

$f^{(4)}(x) = k(k-1)(k-2)(k-3)x^{k-4}$

d Coefficients of x are equal to ${}^kC_p$ in each instance.

e $f^{(p)}(x) = p!\binom{k}{p}x^{k-p}$

2 a $f'(x) = u'v + uv'$

b $f''(x) = (uv)'' = (u'v + uv')' = (u'v)' + (uv')' = u''v + u'v' + uv''$

$= u''v + 2u'v' + uv''$

$$f'''(x) = u'''v + 3u''v' + 3u'v'' + uv'''$$

c $(uv)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} u^{(n-i)} v^{(i)}$

d $f^{(4)}(x) = u^{(4)}v + 4u'''v' + 6u''v'' + 4u'v''' + uv^{(4)}$

3 **Conceptual:** Why is finding a pattern for higher order derivatives of a function useful?

Answer (this is the conceptual understanding): By investigating patterns of higher derivatives of a function, useful rules emerge for finding the *n*th derivative of a function that facilitate the analytical process.

4.4 Graphical interpretation of the derivatives

Investigation 16

Conceptual understanding:

Examining the signs of the gradients to the left and right of stationary points help determine the nature of stationary points, and identify the intervals where the function is increasing/decreasing.

1 0

2 negative

3 positive

4 i $x > 0$

ii $x < 0$

5 **Conceptual:** What is the nature of a stationary point if the gradient of the parabola changes from negative to positive in going through the stationary point?

Answer (this is part of the conceptual understanding): The gradient of a parabola at points close to a minimum point change from negative to positive in going through the point.

6 0

7 positive

8 negative

9 **Conceptual:** What is the nature of a stationary point if the gradient of the parabola changes from positive to negative in going through the stationary point?

Answer (this is part of the conceptual understanding): The gradient of a parabola at points close to a maximum point change from positive to negative in going through the point.

10 **Factual:** In the interval where the gradients are negative, is the function increasing or decreasing?

Answer: decreasing

11 **Factual:** In the interval where the gradients are positive, is the function increasing or decreasing?

Answer: increasing

12 i $x < 0$

ii $x > 0$

13 **Conceptual:** How can you identify the intervals on which a function is increasing/decreasing using the first derivative test?

Answer (this is part of the conceptual understanding): Examining rates of change close to stationary points helps you identify intervals where the function increases or decreases.

14 **Conceptual:** How does the first derivative test help in classifying local extrema and identifying intervals where a function is increasing/decreasing?

Answer (this is the conceptual understanding): Examining the signs of the gradients to the left and right of extrema help determine their nature and identify the intervals where the function is increasing/decreasing.

Investigation 17

Conceptual understandings:

The second derivative describes the rate of change of the gradient function (first derivative); therefore at a stationary point an increasing first derivative function represents a minimum while a decreasing first derivative function represents a maximum.

When the 2nd derivative test fails to determine the nature of a stationary point, revert to testing the gradient on each side of the stationary point to test for a point of inflexion.

1 The gradients go from negative (at points which are left of the minimum point), to 0 (at the minimum), to positive (at points which are right of the minimum). The graph of $f'(x)$ is below the x-axis where the gradients of f are negative, and above the x-axis where the gradients of f are positive.

2 positive

3 increasing

4 The gradients go from positive (at points which are left of the minimum point), to 0 (at the minimum), to negative (at points which are right of the minimum). The graph of $f'(x)$ is above the x-axis where the gradients of f are positive, and below the x-axis where the gradients of f are negative.

The sign of the gradient of $f'(x)$ at any value of x is negative.

5 decreasing

6 $f''(x)$ is positive at a local minimum and negative at a local maximum.

7 Students should find that their results from question **6** hold true for the function

$$(x) = 3 + x + \frac{1}{x}.$$

8 **Conceptual:** How is the second derivative useful in classifying local extrema?

Answer (this is a conceptual understanding): The second derivative describes the rate of change of the gradient function (first derivative); therefore at a stationary point an increasing first derivative function (that is, a positive second derivative) represents a minimum while a decreasing first derivative (that is, a negative second derivative) function represents a maximum.

9 No, since the 2nd derivative is 0, but the stationary point is a minimum point.

10 **Conceptual:** How can you determine the nature of a stationary point when the second derivative test is inconclusive?

Answer (this is a conceptual understanding): When the 2nd derivative test fails to determine the nature of a stationary point, revert to testing the gradient on each side of the stationary point to test for a point of inflexion.

International-mindedness

The Greeks' mistrust of zero meant that Archimedes' work did not lead to calculus.

Further thoughts: Archimedes, one of the greatest ancient Greek mathematicians of all times.

Archimedes was a Greek ancient mathematician, astronomer, physicist, inventor, and engineer. He is credited with introducing infinitesimals, the foundation of calculus.

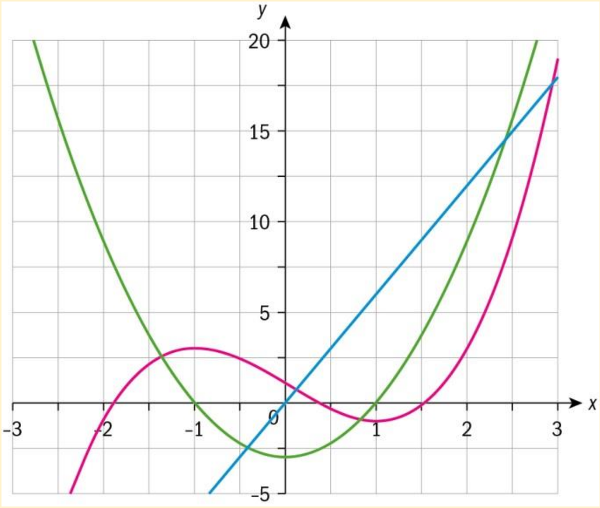
His work was not considered valid as the ancient Greeks did not have zero in their counting system and had a general mistrust of the number.

Investigation 18

Conceptual understanding:

Examining the signs of the second derivative at a stationary point of f' can help identify the intervals where the function is concave up/down since the second derivative indicates if the first derivative is increasing/decreasing.

1 $f'(x) = 3x^2 - 3; f''(x) = 6x$



2 x=0; gradient of f' at x=0 is 0

3 No

4 $f''(0) = 0$

5 Concave down to the left of $x = 0$ and concave up to the right of $x = 0$.

6 negative to the left of $x = 0$ and then positive to the right of $x = 0$.

7 Students should conclude that to the left of $x = 0$, $y = -f(x)$ is concave up and $f''(x)$ is positive; whilst to the right of $x = 0$, $y = -f(x)$ is concave down and $f''(x)$ is negative.

8 **Conceptual:** Why does the sign of a function's second derivative at any point indicate the concavity of the function at that point?

Answer (this is the conceptual understanding): Examining the signs of the second derivative at a stationary point of f' can help identify the intervals where the function is concave up/down since the second derivative indicates if the first derivative is increasing/decreasing.

9 $f''(c) = 0$ at the point of inflexion.

10 Using their results from question 11, students should conjecture that $f''(x) = 0$ at each point of inflexion.

11 No. For $f(x) = x^4$, it is true that $f''(0) = 0$, but this is a minimum point, not a point of inflexion.

12 **Conceptual:** Analyze the concavity of $f(x) = x^3 - 3x + 1$ on both sides of the point of inflexion, and the concavity of $f(x) = x^4$ on both sides of its stationary point. Can you determine the additional condition necessary for *f* to have a point of inflection at $x = c$?

> **Answer:** In addition to $f''(c) = 0$, $f''(x)$ must change sign as *x* passes through *c*. This indicates a change in concavity.

Investigation 19

Conceptual understanding:

At a horizontal point of inflexion *c*, $f'(c) = f''(c) = 0$ and the concavity of *f* changes as *x* passes through *c*.

1 The concavity changes at x=0 and $f''(0) = 0$.

2 Ans. $f'(0) = 0$

3 Positive

4 The signs of the gradients change in going through a local extrema.

5 A point of inflexion whose gradient is parallel to the x-axis.

6 **Conceptual:** What three conditions are necessary for a function $f(x)$ to have a horizontal point of inflexion at $x = c$

> **Answer (this is the conceptual understanding):** At a horizontal point of inflexion *c*, $f'(c) = 0, f''(c) = 0$ and the concavity of *f* changes as *x* passes through *c*.

TOK

The Nature of Mathematics: Does the fact that Leibnitz and Newton came across the Calculus at similar times support the argument of Platonists over Constructivists?

> **Answer:** Platonists would say that mathematical objects exist independently of the human mind and are, thus, discovered. This is often claimed to be the view most people have of numbers.
>
> This objective existence, however, does not mean an empirical existence but, rather, an abstract existence, hence its 'Platonic' label.
>
> The main problem with a Platonist view of mathematics is in the wording. If mathematical objects are abstract objects and objectively exist - then how do we know anything about them?
>
> Constructive mathematics requires that proof be algorithmic. The emphasis in constructive theory is placed on hands-on provability, instead of on an abstract notion of truth.
>
> Note that these are not the only two philosophies of mathematics.

Investigation 20

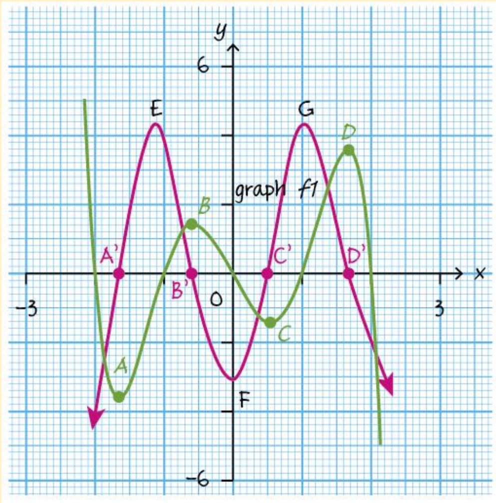
Conceptual understanding:

Identifying turning points, points of inflexion, intervals where the function increases/decreases, and concavity, facilitates sketching the derivatives of the function.

3 i $-2 < x < -1$; $1 < x < 2$ f' is positive

ii $-\infty < x < -2$; $-1 < x < 1$; $2 < x < +\infty$, f' is negative.

4

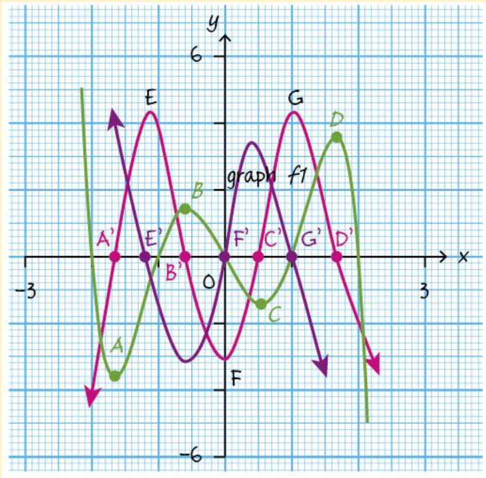


5 Points of inflexion

6 i Concave up: (-∞,-1), (0,1) $f''>0$

ii Concave down: (-1,0), (1,2) $f''<0$

7



8 **Conceptual:** Describe how you can sketch the graphs of $y = f'(x)$ and $y = f''(x)$ from the graph of $y = f(x)$.

Answer (this is the conceptual understanding): Identifying turning points, points of inflexion, intervals where the function increases/decreases, and concavity, facilitates sketching the derivatives of the function.

Investigation 21

Conceptual understanding:

Identifying the zeros of a derivative function, and examining intervals where the function is above/below the x-axis, and where it is concave up/down all facilitate sketching the function from the graph of its derivative.

2 Stationary points of f occur when $f'(x) = 0$; happens when x = 0, 3.

3 $f(x)$ is increasing when $f'(x) > 0$, and $f(x)$ is decreasing when $f'(x) < 0$;

i $]3, \infty[$

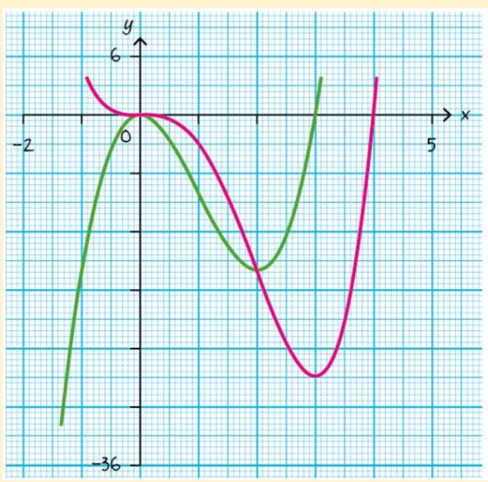
ii $]-\infty, 0[\cup]0, 3[$

4 $f(x)$ is concave up when $f'(x)$ is increasing (that is equivalent to when $f''(x)$ is positive) and $f(x)$ is concave down when .. is decreasing (that is equivalent to when $f''(x)$ is negative).

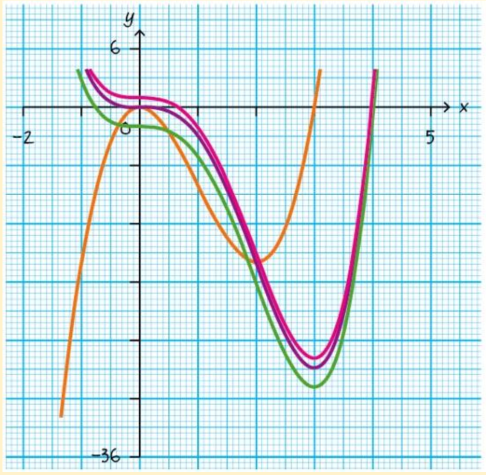
i $]-\infty, 0[$ and $]2, \infty[$

ii $]0, 2[$

5



6



They are all vertical translations of one another.

7 **Conceptual:** Given the graph $y = f'(x)$, describe how can you sketch a possible graph of $y = f(x)$.

Answer (this is the conceptual understanding): Identifying the zeros of a derivative function, and examining intervals where the function is above/below the x-axis, and where it is concave up/down all facilitate sketching the function from the graph of its derivative.

Investigation 22

Conceptual understanding:

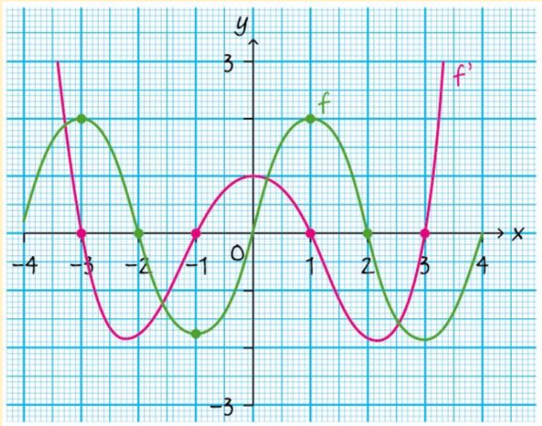
The zeros of f' occur where f'' has stationary points. Where the graph of f'' goes from positive to negative, f' has a maximum, and where the graph of f'' goes from negative to positive f' has a minimum.

1 f has possible points of inflexion where the graph of f'' is 0.

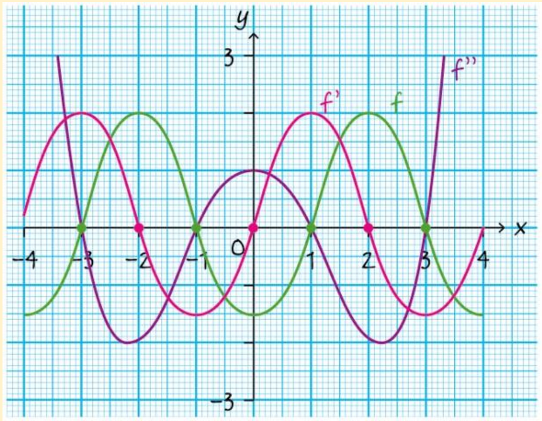
2 **Conceptual:** What can you deduce about the zeros, maximums and minimums of $f'(x)$ from the graph of $y = f''(x)$?

Answer (this is the conceptual understanding): The zeros of f' occur where f'' has stationary points. Where the graph of f'' goes from positive to negative, f' has a maximum, and where the graph of f'' goes from negative to positive f' has a minimum.

3



4



4.5 Applications of differential calculus

TOK

How can you justify the raise in tax for plastic containers e.g. plastic bags, plastic bottles etc. using optimization?

Answer: An important environmental concern that is well document in the media where students might research and write a report using a mathematical model which would allow them to access the skills of mathematical presentation, communication and personal engagement with areas such as the countries where customers have to pay for plastic supermarket bags or a tax increase on water sold in plastic bottles.

Investigation 23

Conceptual understanding:

Optimization in Calculus uses mathematical models, or functions, to provide largest and least-value solutions to real-life problems.

1 $A = 2\pi r^2 + 2\pi rh$; radius and height of the cylinder

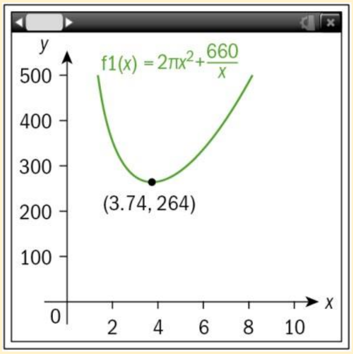
2 $V = \pi r^2 h = 330$

3 ..; $r>0$

4 $\frac{dA}{dr} = 4\pi r - \frac{660}{r^2} = 0$ for min

Solving gives r=3.74cm; h=7.49cm; A=264cm^2

5



6 **Conceptual:** What is optimization in Calculus?

Answer (this is the conceptual understanding): Optimization in Calculus uses mathematical models, or functions, to provide largest and least-value solutions to real-life problems.

7 E.g., cola can: r=6.63cm; h=11.5cm; A=376cm^2

8 Some considerations might be: average hand-size of soft drink consumers, stacking costs on shelves, cost of buying aluminium in bulk, desired aesthetics of can, etc.

Investigation 24

Conceptual understanding:

Finding a mathematical model for a general case allows for multiple applications to specific cases.

1 Student’s own diagram.

2 $A = x(10 - x)$; domain $0 < x < 10$. $x > 0$ because width cannot be negative, and $x < 10$ in order that the length of the enclosure is non-zero.

3 length = width = 5m; Area = 25m^2

4 The largest enclosure is a square.

5 If width = x, then length = $\frac{P - 2x}{2}$ and hence $A = \frac{1}{2}x(P - 2x)$. Differentiating this:

$$\frac{\text{d}A}{\text{d}x} = \frac{P}{2} - 2x.$$

Setting equal to zero: $\frac{\text{d}A}{\text{d}x} = 0 \Rightarrow x = \frac{P}{4}$. Hence, all sides of the rectangle are equal, therefore the rectangle is a square.

6 Letting ***x* be the base of the triangle gives** 19.2cm^2; equilateral triangle.

7 i 688cm^2

ii 866cm^2

8 Students will have different equivalent formulae.

9 As the sides of the regular polygon increase, the figure approaches a circle, so a circle will have the maximum area possible using 20m of border.

10 Students will use the formula they have for **8**, and substitute x for 20m.

11 Knowing the general formula allows you to find the area of any regular polygon if you know its perimeter.

12 **Conceptual:** How does finding a mathematical model for a general case help you apply solutions to particular cases?

Answer (this is the conceptual understanding): Finding a mathematical model for a general case allows for multiple applications to specific cases.

TOK

Read these statements

- Mathematical truths are practical simplifications based on a number of experiences.
- Mathematics is analytical, it is true by definition.
- Math gives us knowledge independent of experience.

Which do you agree with the most? Why?

Answer: This is an opportunity for a written exercise, a blog post or a class debate.

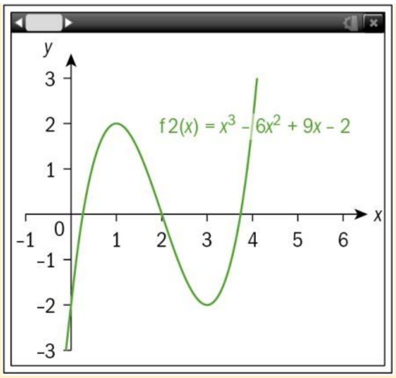
Investigation 25

Conceptual understanding:

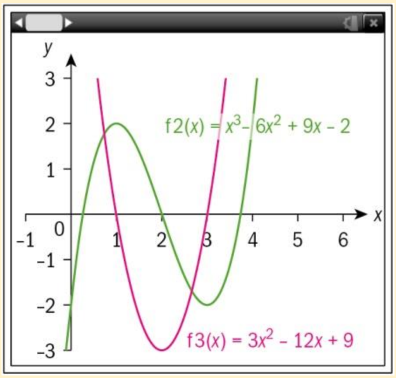
An object moving in a positive direction has positive velocity; an object moving in a negative direction has negative velocity.

Speed is the absolute value of velocity.

1



2



a $0 < t < 1$, Ben is moving from *A* to *B*; $3 < t < 4$, Ben is again moving from *A* to *B*

b $1 < t < 3$, Ben is moving from *B* to *A*

3 Conceptual: What does a positive or negative value for velocity represent?

Answer (this is the conceptual understanding): An object moving in a positive direction (when Ben is running from *A* to *B* in this case) has positive velocity, and an object moving in a negative direction (when Ben is running from *B* to *A* in this case) has negative velocity.

4 Conceptual: What is the connection between speed and velocity?

Answer (this is the conceptual understanding): Speed is the absolute value of velocity.

5 At $t = 1$ Ben has reached *B* and so is momentarily stopped as he turns around; at $t = 3$ Ben has reached *A* and so is momentarily stopped as he turns around.

6 At $t = 0$ and $t = 4$ Ben reaches maximum velocity of 9 m s^{-1}. Ben was already running at 9 m s^{-1} when he passed point *A* for the first time, and he is also running at 9 m s^{-1} when he passes point *B* for the final time.

Investigation 26

Conceptual understanding:

Both positive or both negative velocity and acceleration indicate an increasing speed of the object since they describe the same direction. Velocity and acceleration with different signs indicate a slowing down since they exhibit opposite directions.

1 $v(t) = 3t^2 - 14t + 11; a(t) = 6t - 14$

2 a $t = 1, 3.67$; The graph of speed against time cuts the *x*-axis at these points.

b $1 < t < 2.33$; both acceleration and velocity are negative in this time interval, so speed is increasing in negative direction.

$t > 3.67$; both acceleration and velocity are positive in this time interval, so speed is increasing in positive direction.

c $0 < t < 1$; velocity is positive but acceleration is negative, so the particle's speed is decreasing as it travels in the positive direction.

$2.33 < t < 3.67$; velocity is negative but acceleration is positive, so the particle's speed is decreasing as it travels in a negative direction.

3 **Conceptual:** What must be true about the signs of the velocity and acceleration in order for a particle to speed up, and in order for a particle to slow down?

Answer (this is the conceptual understanding): Both positive or both negative velocity and acceleration indicate an increasing speed of the object since they describe the same direction. Velocity and acceleration with different signs indicate a slowing down since they exhibit opposite directions.

4 $t = 1$, $t = 3.67$; the particle's speed changes sign at these points.

4.6 Implicit differentiation and related rates

Investigation 27

Conceptual understanding:

It is not always possible to change an implicit relation to an explicit one, so you need another method to differentiate implicit functions.

1 $x^2 + y^2 = 1$

2 $y = \pm\sqrt{1-x^2}$ **;** $\frac{dy}{dx} = \pm\frac{x}{\sqrt{1-x^2}}$

3 ±0.577; For $y = -\sqrt{1-x^2}$ **,** $\frac{dy}{dx} = \frac{2x-2y-1}{2x-2y+1}$. For $y = -\sqrt{1-x^2}$, $\frac{dy}{dx} = 0.577$.

4 No, as it is not a one-to-one function. Its graph does not pass the vertical line test.

5 **a** $y = \frac{4-3x}{2}$

b $y = \frac{1-4x}{3+3x}$

c $y = \sqrt{2-x}$

d Cannot solve for y.

6 **Conceptual:** Is it always possible to change an implicitly defined relation into an explicitly defined relation? What do you need in order to find the derivative of any implicitly defined relation?

Answer (this is the conceptual understanding): It is not always possible to change an implicit relation to an explicit one, so you need another method to differentiate implicit functions.

7 $-\infty < x < 1.5$; $0 < x < 1.5$; $0 < x < 1.5$ (Other answers are possible.)

TOK

Mathematics and Knowledge Claims: Euler was able to make important advances in mathematical analysis before Calculus had been put on a solid theoretical foundation by Cauchy and others. However, some work was not possible until after Cauchy's work.

What does this suggest regarding intuition and imagination in Mathematics?

Answer: You might want to consider a few different ways of knowing with statements such as:

Reason and imagination are equally important to understanding mathematics. If one can only visualize it in the head but cannot express it satisfactorily, then the conclusion is flawed and inconsistent.

Much work can also be done using intuition if one has an ingenious insight of the material. For instance, Einstein knew the big ideas of his general theory of relativity but he just lacked the necessary mathematical language to present it until he was helped by mathematicians such as Sir Arthur Eddington.

Reflect: Why is it necessary to have a method for implicit differentiation?

Answer: Differentiation of implicitly defined relations provides a differentiation technique for relations that cannot be expressed explicitly.

TOK

How do we choose the axioms underlying mathematics?

Is this an act of faith?

"Mathematics is the language with which God wrote the Universe" – Galileo

Answer: You might want to define an axiom as a rule or a statement that is accepted as true without proof. An axiom is also called a postulate. An example would be:

"If x and y are real numbers, then $x + y$ is also a real number."

A claim to be investigated might be from Bjarne Stroustrup (a Danish computer scientist) "an axiom is something we can't prove. It is something we assume to be true."

A counterclaim could be the definition of faith from the Merriam-Webster dictionary "firm belief in something for which there is no proof".

Investigation 28

Conceptual understanding:

"Related rates" problems analyse the effects that the change in rate of one variable has on the change in rate of another variable.

1 Student's own diagram.

2 $\frac{r}{h} = \frac{3}{10} \Rightarrow r = 0.3h$

3 $V = 0.03\pi h^3$

4 $\frac{dV}{dt} = 0.09\pi h^2 \frac{dh}{dt} = 2\text{ W}$

$$\Rightarrow \frac{dh}{dt} = \frac{2}{0.09\pi h^2}$$

5 $\frac{dh}{dt} = \frac{1}{0.18\pi}$ cm s^{-1} or 1.77 cm s^{-1}

6 **Factual:** Summarize your findings on how the rates of change of the variables in this problem are related.

Answer: The radius and height of the cone increase proportionately as the volume increases.

7 **Conceptual:** What do "related rates" problems analyse?

Answer (this is the conceptual understanding): 'Related rates' problems analyse the effects that the change in rate of one variable has on the change in rate of another variable.

Factual: Can you think of some other real-life related rates problems?

Answer: There are many, and students should try to list two or three. Make sure they explain how each quantity they list is itself a rate of change.

Modelling and Investigation Activity: River Crossing

Approaches to Learning: Thinking Skills: Evaluate, Critiquing, Applying

Exploration Criteria: Personal engagement (C), Reflection (D), Use of mathematics (E)

IB Topic: Differentiation, Optimization

This task introduces students to the idea that their inspiration for an exploration can come from many different sources - one of which, in this case, may be examples and questions in textbooks. However, if they use these then they are advised to contextualize the idea in real-life in order to score well in Criterion C - Personal engagement. They also need to reflect on the occasional necessary simplifications, assumptions, guesses or estimates that are required in order to model and solve a complicated situation with multiple, and often unknowable, variables (Reflection – Criterion D). The task also gives students the opportunity to practice another optimization problem and demonstrate their understanding of the mathematics of this (Use of mathematics – Criterion E).

It is possible to use textbook examples and questions from exercises as inspiration or springboards for an exploration idea. If a question resonates with a student or reminds them of a particular experience or can be adapted, then it may be possible to develop it into a workable exploration idea.

The problem

The problem is similar to question 12 in Exercise 4S.

To start, you could discuss the problem as a whole class.

Students should start by answering the 'textbook' question given, using the optimization techniques learnt in this chapter.

Visualize the problem

Encourage students to always sketch a diagram to visualize a written mathematical problem. Remind them of the importance of labelling their diagram carefully.

You start at *A*.

B is directly opposite you, on the other bank of the river.

C is a point on the opposite bank *x* km from *B*.

D is the campground.

Solve the problem

Using Pythagoras' theorem $AC = \sqrt{x^2 + 1}$

Remind students of the kinematic formulae that they met in the chapter:

If travel is at a constant rate of speed then

$$\text{Time taken} = \frac{\text{Distance travelled}}{\text{Speed}}$$

Time taken to swim from A to C = $\dfrac{\sqrt{x^2 + 1}}{3}$

Time taken to run from C to D = $\dfrac{2 - x}{8}$

Total time taken $T = \dfrac{\sqrt{x^2 + 1}}{3} + \dfrac{2 - x}{8}$

To find $\dfrac{\mathrm{d}T}{\mathrm{d}x}$, students should first simplify the expression for T:

$$T = \frac{\sqrt{x^2 + 1}}{3} + \frac{2 - x}{8}$$

$$= \frac{1}{3}\left(x^2 + 1\right)^{\frac{1}{2}} + \frac{1}{4} - \frac{x}{8}$$

They can then differentiate:

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{1}{3} \times \frac{1}{2}\left(x^2 + 1\right)^{-\frac{1}{2}} \times 2x - \frac{1}{8}$$

$$= \frac{x}{3\sqrt{x^2 + 1}} - \frac{1}{8}$$

They can find the minimum time taken by setting $\dfrac{\mathrm{d}T}{\mathrm{d}x}$ equal to zero:

$$\frac{x}{3\sqrt{x^2 + 1}} - \frac{1}{8} = 0$$

$$\frac{x}{3\sqrt{x^2 + 1}} = \frac{1}{8}$$

$$8x = 3\sqrt{x^2 + 1}$$

$$64x^2 = 9\left(x^2 + 1\right)$$

$$64x^2 = 9x^2 + 9$$

$$55x^2 = 9$$

$$x^2 = \frac{9}{55}$$

$x = \pm\dfrac{3}{\sqrt{55}}$

Now $x \neq -\dfrac{3}{\sqrt{55}}$ because x is a length and so cannot be negative.

Therefore $x = \dfrac{3}{\sqrt{55}} = 0.405\text{km}$

To evaluate the validity of the value:

The value is less than 2. It is positive.

Using the second derivative test to show that this is a minimum value:

$$\frac{d^2T}{dx^2} = \frac{3(x^2+1)^{\frac{1}{2}}.1 - x.\frac{3}{2}.2x(x^2+1)^{-\frac{1}{2}}}{9(x^2+1)}$$

$$= \frac{(x^2+1)^{-\frac{1}{2}}\left[3(x^2+1) - 3x^2\right]}{9(x^2+1)}$$

$$= \frac{3}{9(x^2+1)^{\frac{3}{2}}}$$

$$= \frac{1}{3(x^2+1)^{\frac{3}{2}}}$$

at $x = \dfrac{3}{\sqrt{55}}$

$\dfrac{d^2T}{dx^2} > 0$ therefore a minimum

For $x = 2$:

$$T = \frac{\sqrt{\left(\frac{3}{\sqrt{55}}\right)^2 + 1}}{3} + \frac{(2 - \frac{3}{\sqrt{55}})}{8} = 0.360 + 0.199 = 0.559\ (3\text{sf})$$

Therefore, the minimum time is 0.559 hours (= 33.5 minutes)

You would swim from point *A* to a point *C* which is 0.405 km from *B*. From there you would run the remaining 1.595km to the campground. The swim would take 0.360 hours (21.6 minutes) and the run would take 0.199 hours (11.9 minutes).

Assumptions made in the problem

Assumptions could include:

The river is exactly 1 km wide.

The campground is exactly 2 km along the bank on the over side of the river.

it is possible to swim at a constant speed of 3 km/h.

The river flow does not push you downstream or make it more difficult to maintain a speed in the middle.

There is no time taken to enter and leave the river.

It is possible for you to run at a constant speed of 8 km/h - over any terrain - for the remaining distance.

The banks of the river are perfectly parallel and perfectly straight.

When you are standing at the edge of the river, you are unlikely to know:

The width of the river.

The distance to the campground on the other side.

The speed you can swim or run.

Encourage students to explore and discuss the problem and their answers to the questions.

You could ask:

What methods could you use to find this information?

How accurate would any distance measurements be?

What assumptions would you need to make when calculating any speeds?

Possible methods for finding distances are:

It is possible to use trigonometry or similar triangles if you have instruments to measure distances and angles.

As an **extension**, you could guide students to research these methods.

It may also be possible to measure on a map if it is a reliable scale, or use GPS if it is available!

The other method is to guess or estimate based on a known distance.

The answers will have varying degrees of accuracy depending on the accuracy of the measuring instruments used or the ability to guess accurately.

Calculations of speed are most likely to be based on a known running/swimming time for a given distance.

The assumptions are that this time remains constant regardless of river flow or terrain.

Additional information you would need to know to determine the shortest time possible:

The effect of the flow of the river.

What you are carrying and whether this will have any effect.

Emphasize to students that in order to make it possible to answer the question in 'real-life' certain assumptions, guesses and estimates do have to be made otherwise the question is too complicated to answer as it has too many unknowns and variables.

However, if this were an exploration it would be important to reflect critically on any assumptions made and the subsequent significance and limitations of the results.

Extension

In this chapter students have been introduced to some classic optimization problems in the examples and exercises. For example, there are 'closed-box' and 'open-box' problems on pages 280 (Example 34) and 281 (Exercise 4R question 5), the rectangular region problem (Investigation 24; Exercise 4R question 1), the cylindrical can problem (Investigation 23) and/or others of the questions in Exercise 4R on page 281.

One problem with the open box problem is that it does not provide the box with any extra material to build stability into the construction. A box built in such a way would probably not be strong and so It would be wise to have tabs on the sides being folded that could be used to add this stability.

In both the open-box problem and the can problem the question of aesthetics is not considered. Also, functionality is not considered. Cans for example are not always built in such a way to provide the maximum volume for the minimum use of materials but also with design, packaging and functionality (e.g. considering average grip size or the need for can openers, etc) in mind.

This could produce an interesting discussion on design and mathematics, etc.

5 Analysing data and quantifying randomness: statistics and probability

Essential understandings

Statistics is concerned with the collection, analysis and interpretation of data and the theory of probability can be used to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events. Statistical representations and measures allow us to represent data in many different forms to aid interpretation.

Probability enables us to quantify the likelihood of events occurring and so evaluate risk. Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned to differentiate between the theoretical and the empirical/observed. Probability theory allows us to make informed choices, to evaluate risk, and to make predictions about seemingly random events.

Content specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Organizing, representing, analysing and interpreting data and utilizing different statistical tools facilitates prediction and drawing of conclusions.
- Different statistical techniques require justification and the identification of their limitations and validity.
- Approximation in data can approach the truth but may not always achieve it.
- Some techniques of statistical analysis, such as regression, standardization or formulae, can be applied in a practical context to apply to general cases.
- Modelling through statistics can be reliable, but may have limitations.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
Different sampling techniques can be used to give a better representation of the population as a whole.	Investigation 1
Simple random samples are normally distributed about the population mean, so the sample mean is a good estimator for the population mean.	Investigation 2
Bias in the collection of data may skew the results and reducing bias allows a sample to represent the general population.	Investigation 3
Standard deviation is a measure of the average distance of each data point from the mean.	Investigation 5
When a data set increases or decreases by a constant amount, the mean is also increased or decreased by the same constant amount but the standard deviation remains the same. When a set of data is multiplied by a constant amount, both the mean and the standard deviation are multiplied by the same constant amount.	Investigation 6

In some cases $\frac{\sum x^2}{n}$ and μ^2 may be given or easily evaluated and therefore this representation of the standard deviation is more practical.	Investigation 7
Some techniques of statistical analysis, such as linear regression can be applied in a practical context.	Investigation 9
The product moment correlation coefficient represents the strength of the correlation between two variables.	Investigation 10
Modelling through statistics can be reliable but it is important to check for inaccuracies and limitations.	Investigation 11

Syllabus sections covered in this chapter:

- SL4.1*
- SL4.2*
- SL4.3*
- SL4.4*
- SL4.10

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and the development of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	**Animated worked example**	**GDC skills and support**	**Additional exercises**
Page 307: mean, median, mode, range, quartiles and interquartile range.	Page 321: Example 4 Page 342: Example 9	Page 320: Example 3 Page 325: Example 5 Page 337: Example 7 Page 338: Example 8	Pages 313, 333, 346, 355

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 357	Page 360	N/A

5.1 Sampling

Investigation 1

Conceptual understanding:

Different sampling techniques can be used to give a better representation of the population as a whole.

1	**Sampling technique**	**Advantages**	**Disadvantages**
	Beth's suggestion: interview their friends	very easy to conduct	only takes students from one year group, so may not be representative
	Emily's suggestion: interview two people from each year group	chooses a good spread from across the different year-groups	Only gives a very small sample size, so may not be big enough to be representative
	Natasha's suggestion: pick 10 boys and 10 girls	gives an equal spread across the sexes in case there is a difference between males and females.	May not choose equal amounts of people from different year-groups.
	Amanda's suggestion: Use a random number generator to choose a sample	Very fair and systematic	Might not get a good split between males & females, or across the year-groups
	Greg's suggestion: choose every 20th person from an alphabetical list	Very fair and systematic Would give a sample of size 75, which is about the right size.	Might not get a good split between males & females, or across the year-groups

2 Answers may vary. Perhaps the fairest way would be to divide students by year group, and then again by gender. Within those subgroups (boys in year 7, girls in year 7, etc.), you could choose every 20th person.

This would give a good sample size, and an equal proportion of males and females across all year groups within the school.

3 Students' own answers. Alternative methods may include asking everybody in school, or asking the students' council.

4 **Conceptual:** Why must you consider the context of a scenario in order to choose an appropriate sampling technique?

Answer (this is the conceptual understanding): Different sampling techniques can be used to give a better representation of the population as a whole.

International-mindedness

The Kinsey Report – famous sampling issues.

The Kinsey Reports were translated into thirteen languages sold over three quarter of a million copies and have been called some of the most influential scientific books of the 20th century.

Further thoughts: The publication of the Kinsey reports in 1948 and 1950 meant for the for the first time that frank talk about sex is no longer the sole domain of the church and psychoanalysts. Now, everyone could be master of that domain. Martin Gumpet says that the frank talk of the newly published Kinsey Report makes it "an important, useful and honest book."

The Kinsey report is considered, by some, to be an important, useful, and honest book. However, it contains many debatable statements, and there are a number of statistical and technical errors–which this reader can only suspect but which other critics have cited with intense disapproval.

Reflect: Is the taxiing speed of an airplane discrete or continuous? Is the number of airplanes waiting to take off discrete or continuous?

Could data ever be classified as both discrete and continuous? Why is it important to consider the nature of the variable, rather than just the data values themselves, when classifying whether data are discrete or continuous?

Answer: Speed is a continuous variable, and so taxiing speed will always be a continuous variable. Although speed can be rounded to the nearest discrete interval, such as the nearest kilometre per hour, this does not change the nature of speed being a continuous variable.

Number of airplanes will always be a discrete variable.

International-mindedness

Ronald Fisher (1890-1962) lived in the UK and Australia and has been described as "a genius who almost single-handedly created the foundations for modern statistical science". He used statistics to analyse problems in medicine, agriculture and social sciences.

Further thoughts: Who else could be considered "the father of statistics"?

Reflect: Define each of these in terms of the context presented in Investigation 1.

Answer: In Investigation 1, these five key quantities are:

Key quantity	**Example from Investigation 1**
Target population	All students in the school
Sampling frame	All students listed on the school enrolment (Students must be listed on the enrolment in order for them to be picked in the sample. In this example, the target population and the sampling frame should be the same).
Sampling unit	Individual student
Sampling variable	Number of hours spent on homework
Sampling variate	Measured in minutes and classed in half-hour intervals: 0-30, 31-60, 61-90, 91-120, 121-150,151-180

Investigation 2

Conceptual understanding:

Simple random samples are normally distributed about the population mean, so the sample mean is a good estimator for the population mean.

1 **Students' own answers.**

2 Yes; in a simple random sample, every member of the population is equally likely to be chosen

3 Population is the set of all numbers that can be chosen. It is $\{0,1,2,3,4,5,6,7,8,9\}$

Mean of the population is 4.5

4 Means of the three samples are 3, 4, 5

5 **Students' own answers.**

6 When dividing by 5, decimal digit is 2,4,6,8,0

7 ${}^{10}C_5 = 252$

There are 26 possible means.

8

Interval	1.8-2.4	2.6-3.2	3.4-4	4.2-4.8	5.0-5.6-4.8	5.8-6.4	6.6-7.2	Total
Frequency	4	24	59	78	59	24	4	252
Relative frequency	0.016	0.095	0.234	0.310	0.234	0.095	0.06	

9 The relative frequency is the probability of obtaining a sample mean in that interval.

10 mean of sample means = 4.5, which is the same as the population mean.

11 **Conceptual:** By considering your answer to question 10, why can the mean of a sample provide an estimate for the mean of a population?

Answer (this is the conceptual understanding): Simple random samples are normally distributed about the population mean, so the sample mean is a good estimator for the population mean.

TOK

The nature of knowing: is there a difference between information and data?

Answer: A common misconception is to consider them to mean the same thing.

Data is raw, unorganized facts or numbers.

When you organise data or present it in a given context so as to make it useful, it is called information.

For example: the individual students' scores on a test are data but the class mean is information.

Investigation 3

Conceptual understanding:

Bias in the collection of data may skew the results and reducing bias allows a sample to represent the general population.

Notes for the teacher: In questions 1 – 4, the students will design an experiment to collect data from different sampling methods.

Their results will be different, but from better sampling techniques the sample mean will better reflect the population mean.

5 Factual: Which sampling technique produced the closest estimate to the population mean?

Answer: results can vary depending on the samples chosen, all these techniques can be justified as producing a close estimate to the population mean. The simple random sample is most likely to be the closest.

6 Conceptual: Which sampling technique would minimize the bias and best represent the population?

Answer: As the sample mean is an unbiased estimator for the population mean, and each technique produces results which are similar, any technique could be used, however the simple random sample will reduce the bias.

7 Conceptual: How do you reduce bias?

Answer: Increase the randomness of the sampling technique, The best technique for reducing bias in sampling is randomization. When a simple random sample of size n is taken from a population, all possible samples of size n in the population have an equal probability of being selected for the sample.

Answers to questions 6 and 7 leads to the conceptual understanding: Bias in the collection of data may skew the results and reducing bias allows a sample to represent the general population.

Reflect: How many students would you need to survey to find a good estimate of the average time students at your school spend doing homework?

Answer: This depends on the number of students in the school, and the age ranges etc. Students could discuss this. It also leads onto *how* to conduct the survey for it to be best representative of the population. This will be covered next.

Developing inquiry skills

In the opening problem for the chapter, you were given the test scores, out of 10, of 32 students.

- Are the test scores an example of discrete or continuous data?

• Before marking every student's test paper, the teacher wishes to choose a sample of eight that will give her an estimate of the mean average mark for the class.

Describe a suitable sampling method the teacher could use.

Answer: These test scores are discrete data.

The scores are given as a list and are not broken up by age, gender, or any other category. therefore, stratified or quota sampling would not work.

The teacher might use simple random sampling, or to ensure a good spread, they might order the scores in ascending order and use systematic sampling.

5.2 Descriptive statistics

International-mindedness

"Lies, damned lies and statistics" is a popularized quote often attributed to Benjamin Disraeli (1804-81) and possibly developed from a court expression stating that there are liars, damned liars and expert witnesses."
Over the years statistics have been used to promote beliefs and opinions by carefully choosing the analysis and representation of the data.

International-mindedness

The 19th century German psychologist Gustav Fechner popularized the median although French mathematician Pierre Laplace had used it earlier.

TOK

Do different measures of central tendency express different properties of the data?

How reliable are mathematical measures?

Answer: You might want to look at number sets where the mean, mode and median are different, and ask which the best measure is to use.

Consider the responses in terms of the perspectives of the different people.

You might want to consider the readings on reliability in statistics.

International-mindedness

What are the benefits of sharing and analysing data from different countries?

Reflect: The interquartile range is sometimes called the range of the "middle half" of the data. Explain why this is the case.

Answer: Half of the data lies between the first and third quartiles, and this is the 'middle half' of the data.

TOK

Can you justify using statistics to mislead others? How easy is it to be misled by statistics?

Answer: Misleading statistics is often the result of the absence of representative samples, eg. Google flu predictor, US presidential elections in 1936, Literary Digest v George Gallup, Boston "pot-hole".

Reflect

1 How do you analyse the distribution of data using a histogram?

Answer: You can see the approximate shape that the data forms by tracing a curve that runs through the tops of the bars.

2 How do the different class intervals change the shape of the distribution?

Answer: Narrower class intervals tend to give a better representation of the true shape of the distribution.

3 Why do we sometimes use relative frequencies?

Answer: Relative frequency of a particular class interval tells you the probability of obtaining a reading within that class interval. It gives you a value between 0 and 1.

Investigation 4

1 Upon first inspection, the shapes of the two histograms look quite similar as they both loosely fit a normal distribution and the heights of the bars in both histograms are more or less the same.

However, when you look at the scale used for frequency, you can see that the frequency of fish of each length measured from the River Blyth is much greater than the frequency of fish measured from the River Avon.

2 The sample size of fish measured from the River Blyth is greater than the sample size of fish from the River Avon.

3 It is very difficult to compare two data sets accurately when the scales on the frequency axis are different to one another.

TOK

Why have mathematics and statistics sometimes been treated as separate subjects?

Answer: Here, we are looking at the nature of mathematics. Teachers might want to consider the history and relevant "newness" of statistics. The amount of numeracy, tables, formulas and charts put statistics into most school mathematics curricula, but as statistics has become more important, its connections with everyday human sciences, natural science, arts and languages suggest teaching statistics across the curriculum might be more appropriate.

This is a good place for teachers to view the knowledge framework for mathematics.

Developing inquiry skills

1 Find the mean, mode and median of the class test scores from the start of this chapter.

Answer: Mean = 5.25, Median = 5.5, Mode = 7

2 Which of mean, median or mode gives the best indication of the "average" score? Are any two averages roughly equal?

Answer: In this case, either the mean or the median is the best measure. These two measures are roughly equal. The mode is not such a good indication of "average score".

5.3 The justification of statistical techniques

TOK

To what extent can we rely on technology to produce our results?

Answer: The median, and the upper quartile divide the ordered data into four groups with approximately the same number of observations in each group.

How can you do this for a small sample size like 2.5, 3.1, 6?

For small samples, the is no obvious way to do this, and concessions of some sort must be made.

Do the quartiles have any meaning for samples of this size?

There are different algorithms in technology simply because different people have different ideas how to make the compromises. They may have slightly different objectives in mind how to use the quartiles in practice.

Respond to the question "To what extent can we rely on technology to produce our results?"

Investigation 5

Conceptual understanding:

Standard deviation is a measure of the average distance of each data point from the mean.

1 Mean for both 14, no mode, median both 14, range = 8

2 Same values for mean, median and range but clearly the data points are different

3

Data point, x	$x - \bar{x}$
10	−4
16	2
14	0
12	−2
18	4

4 Approximately half of them will be negative as they are below the mean and half of them will be positive as they are above the mean.

5 Either: take the absolute value; or: square the numbers. Both of these will result in positive answers.

6 Squaring each value:

Data point, x	$x-\bar{x}$	$(x-\bar{x})^2$
10	−4	16
16	2	4
14	0	0
12	−2	4
18	4	16

7 Degrees Celsius squared, which really has no meaning.

8 8

9 take the Square root of the value

10 2.82 (3.s.f)

11 **Conceptual:** How does this technique find and describe the dispersion of the data?

Answer: This technique finds the standard deviation, which is a measure of the average distance of each data point from the mean.

11 This value = $\sqrt{\dfrac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n}}$

Investigation 6

Conceptual understanding:

When a data set increases or decreases by a constant amount, the mean is also increased or decreased by the same constant amount but the standard deviation remains the same. When a set of data is multiplied by a constant amount, both the mean and the standard deviation are multiplied by the same constant amount.

1, 2

Data	Mean	Standard deviation (3 s.f.)
1,1,1,1,1	1	0
1,2,3,4,5	3	1.41
2,6,7,5,4	4.8	1.72

3

Data	Mean	Standard deviation (3 s.f.)
4,4,4,4,4	4	0
4,5,6,7,8	6	1.41
5,9,10,8,7	7.8	1.72

The mean is also increased by 3 but the standard deviation remains the same.

4

Data	Mean	Standard deviation (3 s.f.)
3,3,3,3,3	3	0
3,6,9,12,15	9	4.24
6,18,21,15,12	14.4	5.16

The mean is multiplied by 3 and the standard deviation is multiplied by 3.

5 Conceptual: What happens to the mean and standard deviation when we add a constant to a set of data and multiply a set of data by a constant?

Answer (this is the conceptual understanding): When a data set increases or decreases by a constant amount, the mean is also increased or decreased by the same constant amount but the standard deviation remains the same. When a set of data is multiplied by a constant amount, both the mean and the standard deviation are multiplied by the same constant amount.

Investigation 7

Conceptual understanding:

In some cases $\frac{\sum x^2}{n}$ and μ^2 may be given or easily evaluated and therefore this representation of the standard deviation is more practical.

1 $\sigma = \sqrt{\frac{\sum (x-\mu)^2}{n}}$

2 $\sigma = \sqrt{\frac{\sum (x^2 - 2x\mu + \mu^2)}{n}}$

3 $\sigma = \sqrt{\frac{\sum x^2 - \sum 2x\mu + \sum \mu^2}{n}}$

4 $\sigma = \sqrt{\frac{\sum x^2}{n} - \frac{\sum 2x\mu}{n} + \frac{\sum \mu^2}{n}}$

5 $\sigma = \sqrt{\frac{\sum x^2}{n} - 2\mu^2 + \mu^2}$

6 $\sigma = \sqrt{\frac{\sum x^2}{n} - \mu^2}$

7, **8** Answer given

9 Conceptual: Conceptual Why is the alternative form of the standard deviation formula useful?

Answer: In some cases $\frac{\sum x^2}{n}$ and μ^2 may be given or easily evaluated and therefore this representation of the standard deviation is more practical.

TOK

Is standard deviation a mathematical discovery or a creation of the human mind?

Answer: A typical teacher led discussion might go something like this:
Over the centuries people have debated whether mathematics is discovered, or if it is simply invented by the minds of great mathematicians.
What do you think?
If you think it is discovered, where are you looking?
If you think it is invented, why can't a mathematician say that he has 4 times 2 =10?
Now, what about the standard deviation?

Investigation 8

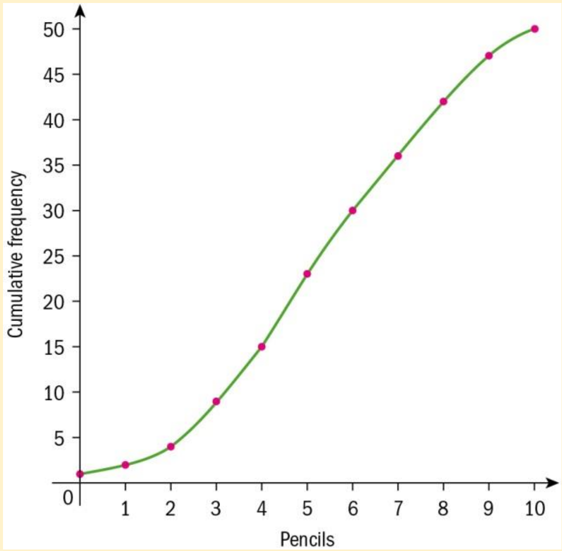
1

Pencils	**Frequency**
0	1
1	1
2	2
3	5
4	6
5	8
6	7
7	6
8	6
9	5
10	3
Total	**50**

2

Pencils	**Frequency**	**Cumulative frequency**
0	1	1
1	1	2
2	2	4
3	5	9
4	6	15
5	8	23
6	7	30
7	6	36
8	6	42
9	5	47
10	3	50
Total	**50**	**50**

3



4 **Factual:** Explain what this graph tells you about how the number of students losing x pencils changes as x increases. Discuss your answer with a classmate.

Answer: An ogive (a cumulative line graph) represents when you want to display the total number of y-values less than or equal to any given x-value.

The relative gradient from point to point will indicate greater or lesser increases; for example, a steeper slope means a greater increase than a more gradual slope.

5 The y-coordinate represents cumulative addition as x increases, so y always increases or stays the same, and hence the curve cannot turn down or to the left.

6 **Conceptual:** How would you use the cumulative frequency curve to find the median and quartiles?

Answer: Go horizontally across to the curve at $y = 12.5$ (for Q_1), and then vertically down to the x-axis to find the lower quartile. Repeat this process at $y = 25$ (for median) and $y = 37.5$ (for Q_3).

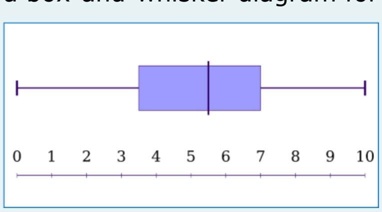
International-mindedness

Ogive originates from the French language as a pointed or Gothic arch. Galton was the first statistician to use it to mean a cumulative frequency curve.

Developing inquiry skills

Return to the opening problem.

1 Draw a box-and-whisker diagram for the data.



2 What does this tell you about the performance of the class?

Answer: The interquartile range is quite small, suggesting that the majority of the class performed at a similar level around the median. There were extremes at either end, too.

5.4 Correlation, causation and linear regression

International-mindedness

In 1956, Australian statistician, Oliver Lancaster made the first convincing case for a link between exposure to sunlight and skin cancer using statistical tools including correlation and regression.

Further thought: The correlation between smoking and lung cancer was "discovered" using mathematics. Science had to justify the cause

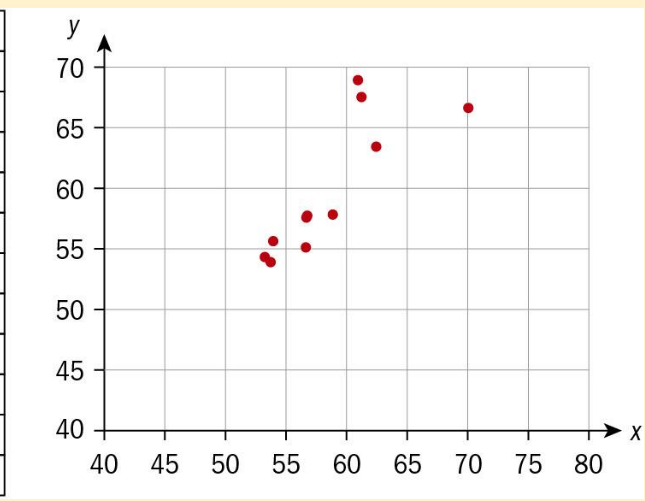
Investigation 9

Conceptual understanding:

Some techniques of statistical analysis, such as linear regression can be applied in a practical context.

1 Construct the scatter diagram. Note that you will not be able to plot a point for student J, since they did not finish run 2.

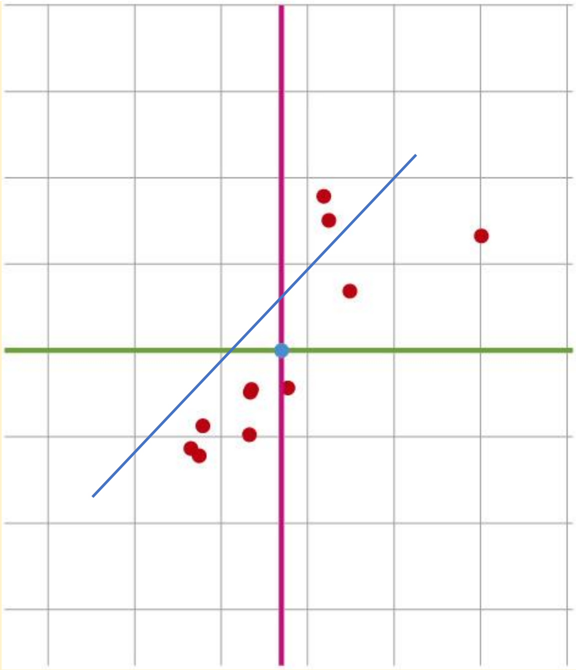
x_1	y_1
53.3	54.3
56.7	57.6
53.8	53.9
54	55.6
61.3	67.5
62.5	63.4
56.7	55.1
58.9	57.8
61.0	68.9
70.1	66.6
56.8	57.7



2 Calculate and plot the mean values for run 1 and run 2

Pupil	A	B	C	D	E	F	G	H	I	J	K	L	total	mean
Run 1	53.3	56.7	53.8	54	61.3	62.5	56.7	58.9	61.0	58.7	70.1	56.8		58.6
Run 2	54.3	57.6	53.9	55.6	67.5	63.4	55.1	57.8	68.9	DNF	66.6	57.7		59.8

3 On your scatter diagram, draw a line of best fit by eye:



4 The equation of the line will be approximately $y = 0.94x + 4.92$

5 Use the table below to ascertain the level of correlation

If majority of points lie in Q1 and Q3	Positive linear correlation
If majority of points lie in Q2 and Q4	Negative linear correlation
If points are evenly distributed or any other situation	No linear correlation

Since most of the points lie in Q1 and Q3, and lie close to the line of best fit, this demonstrates strong positive correlation.

Note: it is important to realise that there may be a correlation that is non-linear.

TOK

How can causal relationships be established in mathematics?

Answer: The first step in establishing causality is demonstrating association; is there a relationship between the independent variable and the dependent variable?

Does the *r* value establish a relationship?

Do intuition and reasoning have parts to play in establishing a mathematical relationship?

Investigation 10

Conceptual understanding:

The product moment correlation coefficient represents the strength of the correlation between two variables.

1

x	y	xy	x^2	y^2
1	3	3	1	9
2	4	8	4	16
4	6	24	16	36
5	8	40	25	64

$n = 4$

$\bar{x} = 3$

$\bar{y} = 5.25$

$\sum xy = 75$

$\sum x^2 = 46$

$\sum y^2 = 125$

2 **Factual:** What is the formula for product moment correlation coefficient?

Answer:

$$r = \frac{\sum xy - n\bar{x}\bar{y}}{\sqrt{\sum x^2 - n(\bar{x})^2}\sqrt{\sum y^2 - n(\bar{y})^2}}$$

In this case, PMCC is

$$r = \frac{75 - 4(3)(5.25)}{\sqrt{46 - 4(3)^2}\sqrt{125 - 4(5.25)^2}}$$

$= 0.988$ (3 s.f.)

3 **Conceptual:** What does product moment correlation coefficient represent?

Answer: The product moment correlation coefficient represents the strength of the correlation between two variables.

International-mindedness

Karl Pearson (1857-1936) was an English lawyer and mathematician. His contributions to statistics include the product-moment correlation coefficient and the chi-squared test.

He founded the world's first university statistics department at the University College of London in 1911.

TOK

Is extrapolation knowledge gained using intuition and, possibly, emotion?

If so, how would you describe interpolation in terms of ways of knowing?

Answer: An opportunity to write a blog post or have a class debate using the ways of knowing.

International-mindedness

On 28th January 1986, millions of people across the world watched the space shuttle *Challenger* break apart just 73 seconds after being launched.

All seven of the crew members aboard perished in the crash, among them was Christa McAuliffe, the first teacher to be invited into an astronaut team.

The cause of this disaster was the failure of an O-ring, which was meant to prevent the hot gases escaping. It is now known that a leading factor for the O-ring failure was the exceptionally low temperature at the time of the launch.

The consideration of the correlation between the external temperature and the erosion of O-rings is now considered on subsequent space launches.

TOK

What is the difference between correlation and causation?

To what extent do these different processes affect the validity of the knowledge obtained?

Answer: Correlation is the idea of modelling a pattern based on data. Causation is using data as proof that one thing causes the other.

Does correlation need causation?

What is a cause and effect relationship?

Is making a model for a given situation valid as personal knowledge?

Investigation 11

Conceptual understanding:

Modelling through statistics can be reliable but it is important to check for inaccuracies and limitations.

1 Using the data in the table with x representing depth of rainfall and y representing % of new bleaching on coral, the line of best fit would be $y = -0.0049136x + 0.75884$.

Hence when x = 78 mm, y = 0.376%.

2 The new prediction would now be 0.690%

3 **Conceptual:** How can you determine if a value is an outlier or unreliable data?

Answer: There are a variety of ways to check for outlier. In this case, researcher 3 has clearly calculate all their percentages incorrectly by a power of 10, and therefore their results should be re-calculated or removed.

This leads to the conceptual understanding: Modelling through statistics can be reliable but it is important to check for inaccuracies and limitations.

Modelling and Investigation Activity: Rank my maths!

Approaches to Learning/learner profile: Collaboration, Communication

Exploration Criteria: Personal engagement (C), Use of mathematics (E)

IB Topic: Bivariate Data, Correlation, Spearman's rank

This task asks students to consider Spearman's Rank as a valid alternative to Pearson's. In explorations students often automatically consider Pearson's correlation because that is the correlation method they have been taught. They will then blindly apply a linear best fit line which is entirely inappropriate given the data under discussion. This shows a lack of mathematical understanding and prevents students from reaching the higher levels in Criterion E: Use of mathematics. By researching and using Spearman's rank as a possibly valid alternative, students can also gain credit in criterion C: Personal engagement, as they will be exploring new mathematics.

There is also the opportunity to design an experiment that looks at student's rankings and to compare these.

Pearson product moment correlation coefficient

Remind students that:

Pearson product moment correlation (PMCC) evaluates the **linear** relationship between two continuous variables. A relationship is **linear** when a change in one variable is associated with a proportional change in the other variable.

In the example, there is clearly a strong relationship between x and y. This is not linear (perhaps exponential).

PMCC is 0.726

This suggests there is a moderate linear correlation.

The relationship is not actually linear, so the PMCC does not show a strong correlation even though a strong relationship exists.

Spearman rank correlation

A more complete definition of the Spearman rank correlation is:

*The Spearman correlation evaluates the **monotonic** relationship between two **variables**. In a monotonic relationship, the variables tend to change together, but not necessarily at a **constant rate**. The Spearman correlation coefficient is based on the **ranked values** for each variable rather than the **raw data**.*

Check that students understand all of the bold words in the given definition.

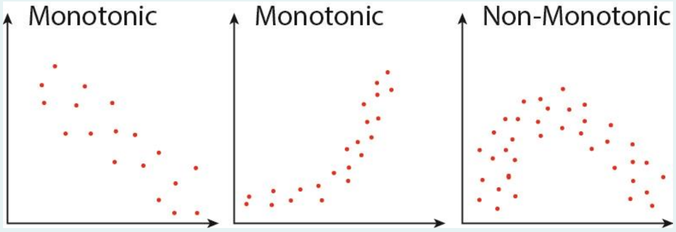
Note: Spearman rank correlation is not on the HL syllabus, but it shows an alternative method.

Monotonic: A monotonic relationship is a relationship that does one of the following:

1) as the value of one variable increases, so does the value of the other variable

2) as the value of one variable increases, the other variable value decreases.

Examples of monotonic and non-monotonic relationships are presented in these diagrams:



This is a good app that allows students to play around with points and see the effect on both the Pearson and Spearman correlations:

https://www.economicsnetwork.ac.uk/statistics/pearson_spearman.htm

As **extension** work, students could investigate how the formula for Spearman's correlation is derived.

Spearman's rank correlation can be used to see the strength and direction of the monotonic relationship between two variables and can therefore be useful in explorations.

For the given data, $r_s = 1 - 0 = 1$

Students may need help to evaluate the value of r_s.

This table shows how to interpret Spearman's rank correlation coefficient.

r_s	Correlation
$r_s = 1$	perfect positive correlation
$0.7 \le r_s < 1$	strong positive correlation
$0.4 \le r_s < 0.7$	moderate positive correlation
$0 < r_s < 0.4$	weak positive correlation
$r_s = 0$	no correlation
$0 > r_s > -0.4$	weak negative correlation
$-0.4 \ge r_s > -0.7$	moderate correlation
$-0.7 \ge r_s < -1$	strong negative correlation
$r_s = -1$	perfect negative correlation

This calculated value is perfect spearman's rank correlation.

The answer will always be between -1 and 1.

As with Pearson, a value close to -1 represents a strong negative rank correlation and near to +1 represents a strong positive rank correlation. A value close to 0 suggests no rank correlation. 1 is a perfect positive monotonic relationship and -1 is a perfect monotonic negative relationship.

Activity 1

It is important that students appreciate that if a scatterplot shows that the relationship between two variables looks monotonic they should run a Spearman's correlation because this will then measure the strength and direction of this monotonic relationship. However, if the relationship appears linear you would run a Pearson's correlation because this would measure the strength and direction of any linear relationship. If you are not able to visually check whether you have a monotonic relationship, you might run a Spearman's correlation anyway.

Pearson's correlation is not appropriate because the relationship is non-linear, negative, strong.

If needed, assist students in following the steps to calculate the Spearman's rank correlation coefficient for this data.

For **extension** work, students could also get the value by putting the **ranks** (rather than the raw data) into a graphical calculator and finding the value of *r* as described in this chapter. This is the value of r_s for the data.

The value is −0.721.

There appears to be a negative correlation between them which is strong. However it is not linear.

Activity 2

Introduce this task by explaining:

Spearman's rank can also be useful in explorations as it gives you the opportunity to design an experiment that could compare the rankings given to something by two people (or two sets of people) to determine how similar they are and what agreement there is.

Arrange students in groups of three if possible (pairs and fours would also work but would mean fewer/more results to analyse). Given the nature of the task it may be worth using randomly picked groups (although using groups that are not random and allowing students to choose groups also gives interesting discussions).

Instruct the groups to select between six and ten different pieces of music.

Students could perhaps choose a random playlist or the first few random songs in someone's music collection, or they could deliberately choose particular songs if they wish to test a particular hypothesis.

Students should rank their songs by number from favourite to least favourite, with 1 being the favourite.

Here is an example of a table that could be used:

Song number	Student 1 rank	Student 2 rank	Student 3 rank
1			
2			
3			
Etc…			

To prompt discussion, you could ask:

Do you expect the rankings within your group to be the same, similar or completely different? Could you predict who might have similar tastes (where the strongest correlation would be)?

Check that the students calculate the Spearman Rank correlations correctly.

When evaluating the correlations, students should consider the size and direction of the values.

Check that students write clear, concise and relevant conclusions.

Extension

Students could think about designing a similar experiment to compare ranks in students taste for films, art, hobbies, food etc.

The experimenter should be careful to not give any indication of bias or preference themselves.

It is important also that there is no discussion between students so that they do not come to an agreement/disagreement but rather choose independently and without undue influence.

Also encourage students to prepare the table/form etc that they will use to records their results before they conduct their experiment. This will also ensure that they have considered the nature of the results that they are going to obtain.

6 Relationships in space: geometry and trigonometry

Essential understandings

Geometry and trigonometry allows us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- The properties of shapes depend on the dimension they occupy in space.
- Volume and surface area of shapes are determined by formulae, or general mathematical relationships or rules expressed using symbols or variables.
- The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.
- Equivalent measurement systems, such as degrees and radians, can be used for angles to facilitate ease of calculation.
- Different representations of the values of trigonometric relationships, such as exact or approximate, may not be equivalent to one another.
- The trigonometric functions of angles may be defined on the unit circle, which can visually and algebraically represent the periodic or symmetric nature of their values.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
The properties of shapes depend on the dimension they occupy in space. Certain properties of 2D geometry (e.g. Pythagoras' theorem) can be extended into 3D geometry.	Investigation 1
Knowledge of calculus is important for understanding the relationship between volume and surface area.	Investigation 2
Relationship between the measure of radians and degrees – searching for patterns and describing rules consistent with findings. One radian can be thought of as the angle made by taking the radius and wrapping it round the circumference of a circle.	Investigation 3
We know from the identity that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, hence when $cos\ \theta$=0 by definition of division by zero $\tan\theta$ is undefined.	Investigation 4

Knowledge of symmetry and trigonometric identities in the unit circle will help to find the coordinates of special angles.	
Trigonometric ratios have non-distributive properties.	Investigation 5
The sine curve is a periodic function which implies that there will always be multiple *x*-values which give the same value for y.	Investigation 6
The amplitude, period and vertical and horizontal translations are key features for representing transformed trigonometric graphs.	Investigation 7
Knowledge of inverse and composite functions along with knowledge of basic trigonometry can be synthesized to find derivatives of inverse trigonometric functions.	Investigation 8

Syllabus sections covered in this chapter:

- SL3.1*
- SL3.2*
- SL3.3*
- SL3.4
- SL3.5
- SL3.6
- SL3.7
- SL3.8
- AHL3.9
- AHL3.10
- AHL3.11

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 368: Relationships in space: geometry and trigonometry	Page 390: Example 10 Page 406: Example 21 Page 407: Example 24 Page 423: Example 34	Page 380: Example 5 Page 392: Example 11 Page 395: Example 14 Page 395: Example 15 Page 407: Example 25 Page 419: Example 31 Page 420: Example 32 Page 426: Example 39 Page 429: Example 41	Pages 377, 383, 408, 420, 432

Assessment opportunities

End of chapter test	Chapter review	Exam practice
Page 433	Page 437	Page 438

6.1 The properties of three-dimensional space

Investigation 1

Conceptual understandings:

The properties of shapes depend on the dimension they occupy in space.

Certain properties of 2D geometry (e.g. Pythagoras' theorem) can be extended into 3D geometry.

1 Find the 3D coordinates of B, C, D, E, F and G.

Answer: (1,1,0) (0,1,0) (0,0,1) (1,0,1) (1,1,1) (0.1,1)

2-3 Answers will vary.

4 **Conceptual:** How do 3D coordinates differ from 2D coordinates?

Answer: 3D coordinates have x, y and z components.

5 Find the coordinates of the point at the centre of the cube.

Answer: (0.5,0.5,0.5)

6 Find the length of OB. Find the length of OF.

Answer: $\sqrt{2}, \sqrt{5}$

7 Answers will vary.

8 **Conceptual:** How does understanding 2D geometry enhance the understanding of 3D geometry.

Answer: Certain properties of 2D geometry (e.g. Pythagoras' theorem) can be extended into 3D geometry.

Investigation 2

Conceptual understanding:

Knowledge of calculus is important for understanding the relationship between volume and surface area.

1 Given that the area of a circle is πr^2, use calculus to find the circumference of a circle.

Answer: $C = \frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$

2 Given that the volume of a sphere is $\frac{4}{3}\pi r^3$, use calculus to find the surface area.

Answer: $A = \frac{\mathrm{d}V}{\mathrm{d}r}$, $4\pi r^2$

3 **Conceptual:** How can this relationship be extended to squares and cubes?

Answer given in investigation.

4 Find the area of this square and the corresponding volume of the cube.

Answer: $A = (2x)^2 = 4x^2$, $V = (2x)^3 = 8x^3$

5 Hence, find the perimeter of the square and the surface area of the cube.

Answer: $P = 8x$, $A = 24x^2$

6 **Conceptual:** What is the relationship between circumference and area in a 2D shape? What is the relationship between surface area and volume in a 3D shape?

Answer: Answers given in investigation.

TOK

Starting by showing this quote from Galileo - "Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed.

"It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth."

Ask students for their response to the quote.

You might want to consider exploring how the Platonic Solids govern the structure of any given atom.

6.2 Angles of measure

Investigation 3

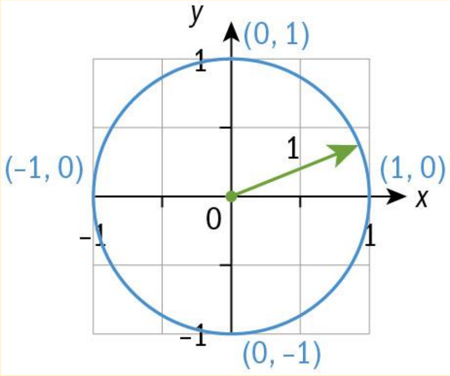
Conceptual understandings:

Relationship between the measure of radians and degrees – searching for patterns and describing rules consistent with findings.

One radian can be thought of as the angle made by taking the radius and wrapping it round the circumference of a circle.

1 Draw a circle with its centre at the origin of a coordinate plane and a radius of 1 arbitrary unit. Label the point A (1,0) on your diagram.

Answer:



2 Find the circumference of this circle in terms of π.

Answer: $C = 2\pi$

3 Complete the following table by finding the length of the following arcs on this unit circle, in terms of π:

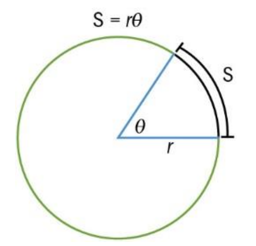
Answer:

Central angle in degrees	**Length of corresponding arc**
360° (circumference)	2π
180°	π
90°	$\frac{\pi}{2}$
60°	$\frac{\pi}{3}$
45°	$\frac{\pi}{4}$
30°	$\frac{\pi}{6}$

The values added in the second column are the radian measures of the angles given in degrees in the first column.

4 **Conceptual:** How do you explain what a radian is?

Answer: One radian can be thought of as the angle made by taking the radius and wrapping it round the circumference of a circle.



5 Consider a circle that has a radius r of any length and an arc of length s along the circle. Complete the following table by finding the length of the following arcs on this unit circle, in terms of r and π:

Answer:

Central angle in degrees	**Length of corresponding arc s**
360° (circumference)	$2\pi r$
180°	πr

90°	$\frac{\pi r}{2}$
60°	$\frac{\pi}{3}r$
45°	$\frac{\pi}{4}r$
30°	$\frac{\pi}{6}r$

6 Use your findings to suggest a formula that describes the relationship between the radius *r* and arc length *s* of the corresponding arc on the circle, if the angle θ is in radian measure.

Answer: $s = r\theta$

7 **Conceptual:** Explain why radians are a *unitless* measure.

Answer: $1 \text{ radian} = \frac{\text{length of arc}}{\text{radius of arc}}$, As both of these are measured in the same unit, they will cancel out to give a ratio with no units.

6.3 Ratios and identities

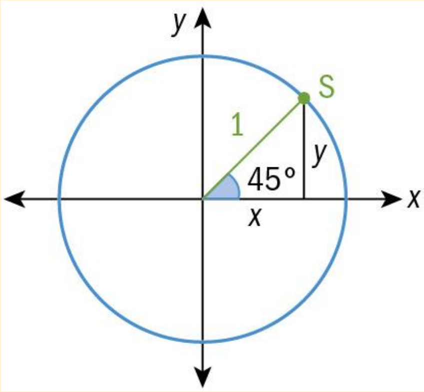
Investigation 4

Conceptual understandings:

We know from the identity that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, hence when $\cos\theta = 0$ by definition of division by zero $\tan\theta$ is undefined.

Knowledge of symmetry and trigonometric identities in the unit circle will help to find the coordinates of special angles.

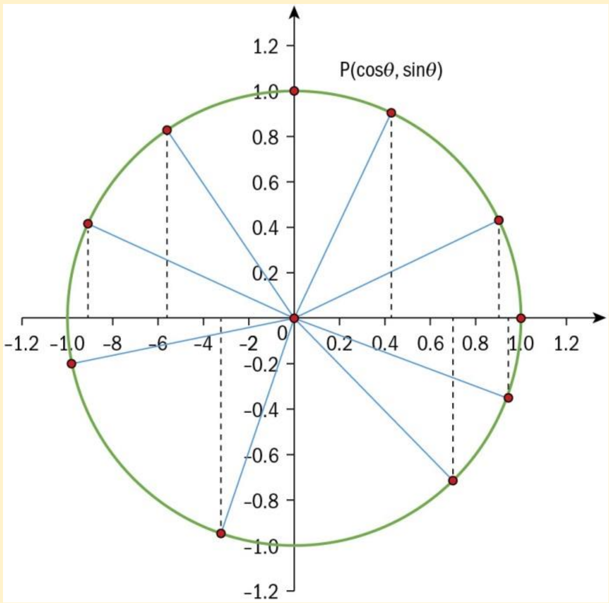
1 Use Pythagoras' theorem to determine the relationship between the *x*- and *y*-coordinates of S. Hence, find the values of *x* and *y*. Leave your answers as simplified radicals.



Answer: $x = \frac{\sqrt{2}}{2}, \ y = \frac{\sqrt{2}}{2},$

2 Take an arbitrary point on the circumference of the circle such that the acute angle is θ. Label this point P. Draw the radius OP and draw a vertical line from P down to the x-axis.

Answer:



3 Use $\sin\theta = \frac{o}{h}, \cos\theta = \frac{a}{h}, \tan\theta = \frac{o}{a}$ to write down the values $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of x and y. Remember, the radius of the circle is 1.

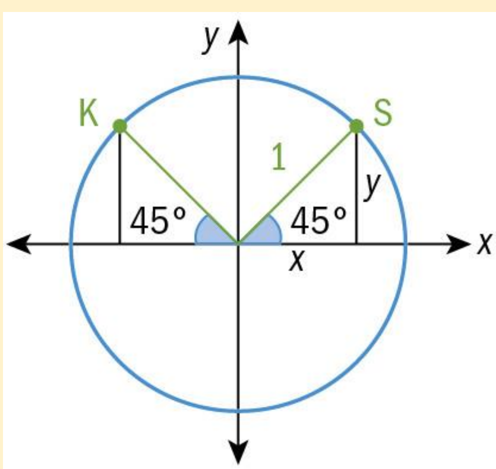
Answer:

$\sin\theta = \frac{y}{1} = y$

$\cos\theta = \frac{x}{1} = x$

$\tan\theta = \frac{y}{x}$

4 Reflect point S in the y-axis so that it appears in the 2nd quadrant, labelling the new point K.



5 Find the angle KÔA in degrees and in radians.

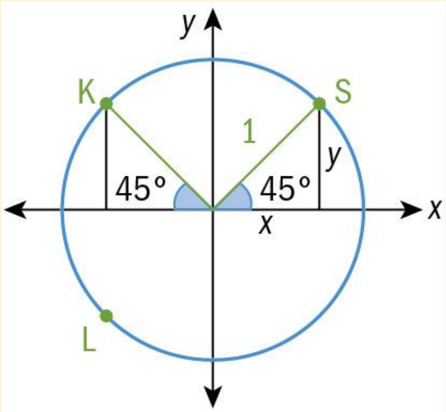
Answer: 135 or $\frac{3\pi}{4}$

6 Write down the coordinates of K in terms of $\sin\theta$ and $\cos\theta$ as simplified radicals. Find the exact value of $\tan\theta$.

Answer: $\sin\theta = \frac{y}{1} = \frac{\sqrt{2}}{2}$, $\cos\theta = -\frac{x}{1} = -\frac{\sqrt{2}}{2}$, $\tan\theta = -\frac{y}{x} = -1$

7 Reflect point K in the x-axis so that it appears in the 3rd quadrant. Label this point L.

Answer:



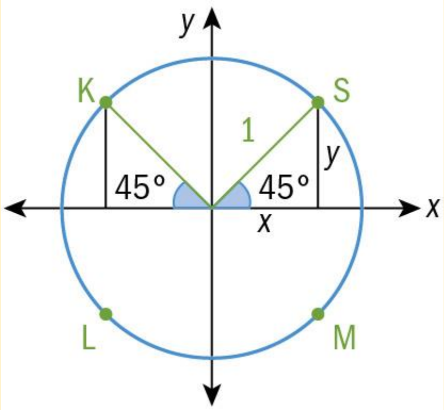
8 Find the reflex angle LÔA in degrees and in radians.

Answer: 225 or $\frac{5\pi}{4}$

9 Write down the coordinates of L in terms of $\sin\theta$ and $\cos\theta$ as simplified radicals. Find the exact value of $\tan\theta$.

Answer: $\sin\theta = \frac{y}{1} = -\frac{\sqrt{2}}{2}$, $\cos\theta = \frac{x}{1} = -\frac{\sqrt{2}}{2}$. $\tan\theta = \frac{y}{x} = 1$

10 Reflect point L in the y-axis so that it appears in the 4th quadrant. Label this point M.



11 Find the reflex angle MÔA in degrees and in radians.

Answer: 315 $\frac{7\pi}{4}$

12 Write down the coordinates of M in terms of $\sin\theta$ and $\cos\theta$ as simplified radicals. Find the exact value of $\tan\theta$.

Answer: $\sin\theta = \frac{y}{1} = -\frac{\sqrt{2}}{2}$, $\cos\theta = \frac{x}{1} = \frac{\sqrt{2}}{2}$, $\tan\theta = \frac{y}{x} = -1$

13 Use your diagram to find in exact form the values of

a cos 30° **b** sin 30° **c** tan 30°

d cos 60° **e** sin 60° **f** tan 60°

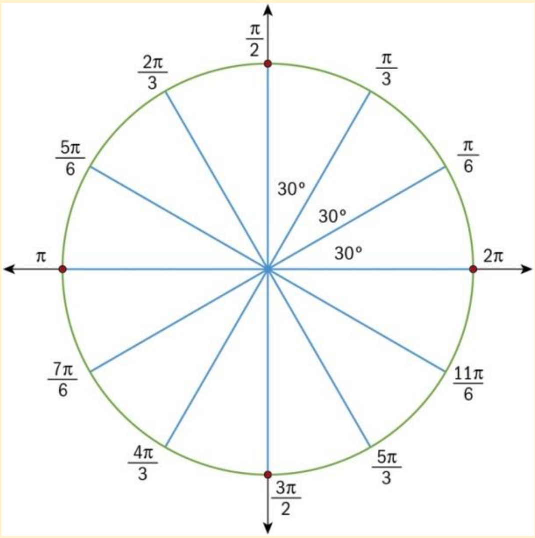
Answers: **a** $\cos 30 = \frac{y}{1} = \frac{\sqrt{3}}{2}$ **b** $\sin 30 = \frac{x}{1} = \frac{1}{2}$

c $\tan 30 = \frac{y}{x} = \sqrt{3}$ **d** $\cos 60 = \frac{y}{1} = \frac{1}{2}$

e $\sin 60 = \frac{x}{1} = \frac{\sqrt{3}}{2}$ **f** $\tan 60 = \frac{y}{x} = \frac{\sqrt{3}}{3}$

14 Sketch a new unit circle with its centre as the origin of a coordinate plane and a radius of 1 arbitrary unit. Sketch two radii that make a 30° angle and a 60° angle from the *x*-axis. Label the points on the circle P1 and P2 respectively.

Answer:



15 Write down the coordinates of P1 and P2.

Answer: P1$(\frac{\sqrt{3}}{2}, \frac{1}{2})$ P2$(\frac{1}{2}, \frac{\sqrt{3}}{2})$

16 Repeat steps for both points P1 and P2: reflect both points in the *x*- and *y*-axes to find the corresponding sin and cos values in quadrants II, III and IV.

Answer:

sin 30	$\frac{1}{2}$
cos 30	$\frac{\sqrt{3}}{2}$
tan 30	$\frac{\sqrt{3}}{3}$
sin 60	$\frac{\sqrt{3}}{2}$
cos 60	$\frac{1}{2}$
tan 60	$\frac{\sqrt{3}}{1}$
sin 120	$\frac{\sqrt{3}}{2}$
cos 120	$-\frac{1}{2}$
tan 120	$-\frac{\sqrt{3}}{1}$

sin 150	$\frac{1}{2}$
cos 150	$-\frac{\sqrt{3}}{2}$
tan 150	$-\frac{\sqrt{3}}{3}$
sin 210	$-\frac{1}{2}$
cos 210	$-\frac{\sqrt{3}}{2}$
tan 210	$\frac{\sqrt{3}}{3}$
sin 240	$-\frac{\sqrt{3}}{2}$
cos 240	$-\frac{1}{2}$
tan 240	$\frac{\sqrt{3}}{1}$
sin 300	$-\frac{\sqrt{3}}{2}$
cos 300	$\frac{1}{2}$
tan 300	$-\frac{\sqrt{3}}{1}$
sin 330	$-\frac{1}{2}$
cos 330	$\frac{\sqrt{3}}{2}$
tan 330	$-\frac{\sqrt{3}}{3}$

17 Summarise your findings in the table.

Answer:

		sin	**cos**	**tan**
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{1}$
90	$\frac{\pi}{2}$	1	0	undefined
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{1}$
135	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1

150	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
180	π	0	-1	0
210	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
225	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
240	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{1}$
270	$\frac{3\pi}{2}$	-1	0	undefined
300	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{1}$
315	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
330	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
360	2π	0	1	0

18 **Conceptual:** Explain why some values in the diagram are undefined.

Answer: We know from the identity that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, hence when $\cos\theta$=0 by definition of division by zero $\tan\theta$ is undefined.

Conceptual: Why are the angles 45, 30 and 60 called special angles?

Answer: Knowledge of symmetry and trigonometric identities in the unit circle will help to find the coordinates of special angles.

Conceptual: Why do different angles lead to the same ratios?

Answer:

TOK

Starting by showing this quote from Galileo - "Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed.

"It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth."

Ask students for their response to the quote.

You might want to consider exploring how the Platonic Solids govern the structure of any given atom.

TOK

In how many ways can you prove Pythagoras' theorem?

TOK

Use of the fact that the cosine rule is one possible generalization of Pythagoras' theorem to explore the concept of "generality".

Investigation 5

Conceptual understanding:

Trigonometric ratios have non-distributive properties.

1 Complete the table using your knowledge of special triangles.

Answer:

sin 30	$\frac{1}{2}$
cos 30	$\frac{\sqrt{3}}{2}$
tan 30	$\frac{\sqrt{3}}{3}$
sin 45	$\frac{\sqrt{2}}{2}$
cos 45	$\frac{\sqrt{2}}{2}$
tan 45	$\frac{1}{1}$
sin 60	$\frac{\sqrt{3}}{2}$
cos 60	$\frac{1}{2}$
tan 60	$\frac{\sqrt{3}}{1}$
sin 90	1
cos 90	0
tan 90	∞

2 Can the angles in trigonometrical ratios be distributed?

Answer: Is sine distributive? No

Does sin 30 +sin 60 =sin 90? No

Does sin 45 +sin 45 = sin 90? No

Does cos 30 + cos 60 = cos 90? No

Does cos 45 + cos 45 = cos 90? No

3 Find an expression for sin 90 using combinations from sin 30, sin 60, cos 30 and cos 60.

Answer: $\sin 30 \cos 60 + \cos 60 \sin 30$

Conceptual: Do the trigonometric functions of compound angles follow the distributive property? Explain your answer.

Answer: Trigonometric ratios have non-distributive properties.

TOK

You might want to consider timelines of discovery of a mathematical value like π or a topic like trigonometry and it will reveal scholars from around the globe that took an idea and built on it and passed it on. An aid would be the Isaac newton quote "If I have seen further it is by standing on the shoulders of Giants."

Investigation 6

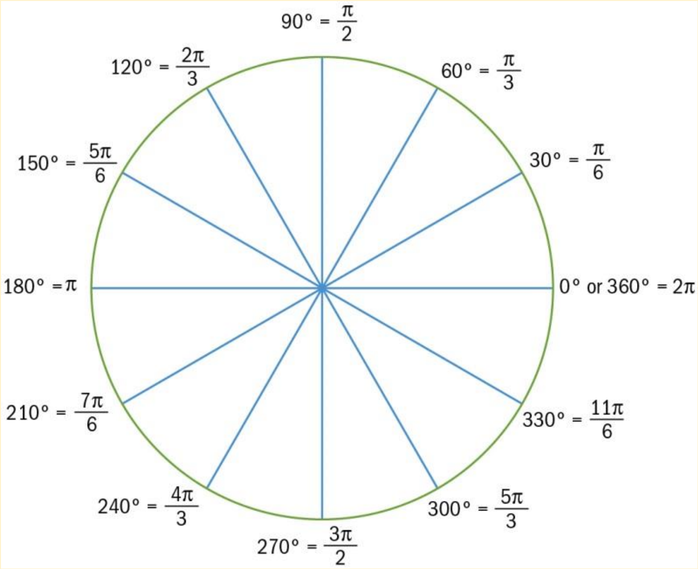
Conceptual understanding:

The sine curve is a periodic function which implies that there will always be multiple x-values which give the same value for y.

1 Answers will vary.

2 Use radian measure to measure and mark every 6π around the circle.

Answer:



3-7 Answers will vary.

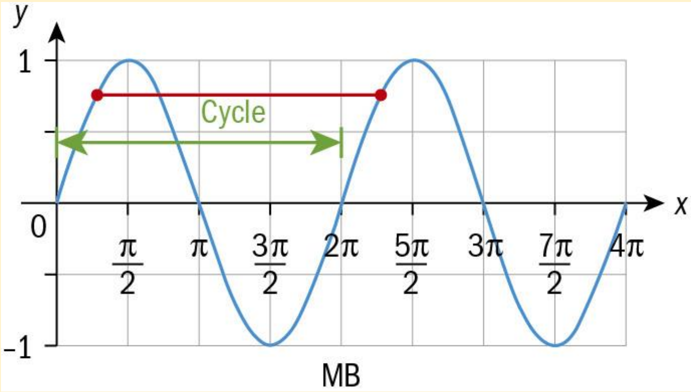
8 **Factual:** What is a periodic function? What shape is the graph of a periodic function?

Answer: A function which repeats itself in periods; the sine curve is periodic because after one revolution the values begin to repeat. The period of a sine curve is 360 degree or 2 pi.

9 Use your calculator to graph $y = \sin x$. Does it look like your spaghetti graph?

Conceptual: How does the shape of a periodic function show there will always be multiple x-values that give the same values of y?

Answer: As the sine curve is periodic there may be multiple values of y that give the same value for y in multiple periods.

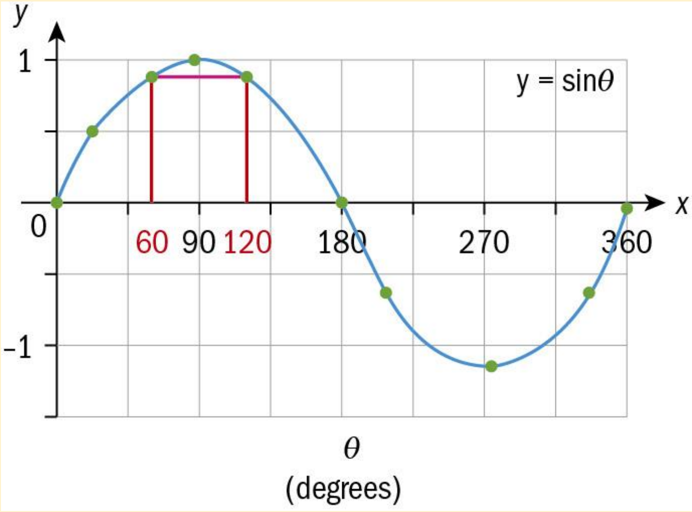


10 Write an explanation about why sin 60° = sin 120°.

Answer: Sin 60 and sin 120 are symmetrically placed on the unit circle and therefore sin 60 = sin 120. Therefore, on the graph they share the same y-value.

Conceptual: How is a sine curve produced?

Answer: The sine curve is a periodic function which implies that there will always be multiple x-values which give the same value for y.



TOK

You might want to consider timelines of discovery of a mathematical value like π or a topic like trigonometry and it will reveal scholars from around the globe that took an idea and built on it and passed it on. An aid would be the Isaac Newton quote: "If I have seen further it is by standing on the shoulders of Giants."

6.4 Trigonometric functions

Investigation 7

Conceptual understanding:

The amplitude, period and vertical and horizontal translations are key features for representing transformed trigonometric graphs.

1 Show that the amplitude of the function is given by |a|.

Answer: $f(x) = a\sin[b(x+c)] + d$

$$a = \frac{\text{maximum value - minimum value}}{2}$$

2 Show that the function has a period of $\frac{2\pi}{b}$.

Answer: The period of the function $f(x) = \sin x$ is 2π. However, when the horizontal stretch is applied the period is $\frac{2\pi}{b}$.

3 Show that the line $y = d$ is a horizontal line of symmetry.

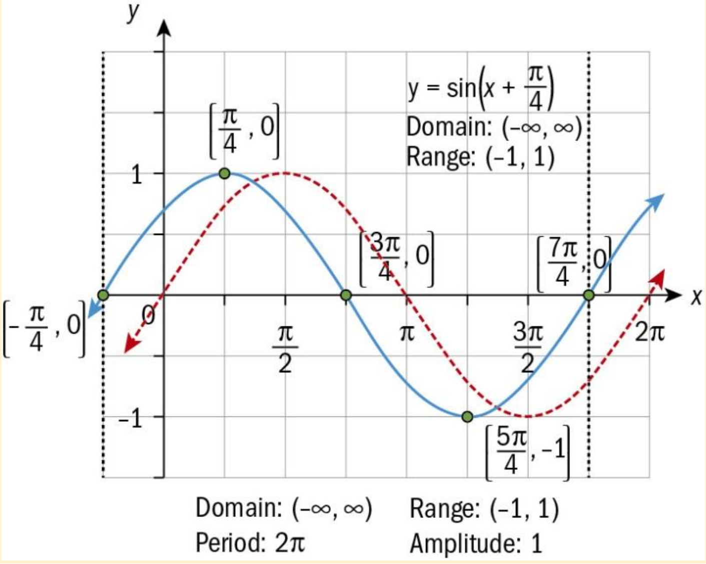
Answer: Strictly speaking, the horizontal line is not a traditional line of symmetry. However, the points where the curve intersects the line are centres of 180 degree rotational symmetry; hence, it is often referred to as a line of symmetry.

4 Show that $d = \frac{\text{maximum value + minimum value}}{2}$.

Answer: The sine curve increases and decrease by a constant amount, and therefore contains rotational symmetry about the points of intersection with the horizontal line $y = d$. This horizontal line of symmetry can be found by finding the midpoint of the maximum and minimum values on the curve $d = \frac{\text{maximum value + minimum value}}{2}$.

5 Show that the function can be obtained by shifting the graph of $y = a\sin(bx)$ by c units to the left and d units vertically upwards when $c, d > 0$.

Answer: Use technology with sliders to show each value separately.



Conceptual: How do we describe the transformation of a trigonometric function?

Answer: The transformations of a trigonometric function can be described fully by determining the constants *a*, *b*, *c* and *d* and understanding the characteristics and effects of these constants.

TOK

Has hearing music ever made you happy or sad? It is said that emotions conveyed by music are a direct result of mathematical relationships in the intervals between the notes.

Mathematics has often been compared with music. For example, in a letter from 1712 to Goldbach, Leibniz remarks that "Music is a hidden arithmetic exercise of the soul, which does not know that it is counting".

Does this mean that music is mathematical? that mathematics is musical or that both are reflections of a common "truth"?

Scales and base eight might also be explored.

An interesting video can be found at https://youtu.be/ufpatylwPrc

6.5 Trigonometric equations

TOK

You might want to consider the effect of graphing a periodic function and a horizontal line and searching for intercepts.

Does an infinite number of solutions lead to an answer? Can you reason what an infinite solution looks like are is this in the realms of imagination?

Investigation 8

Conceptual understanding:

Knowledge of inverse and composite functions along with knowledge of basic trigonometry can be synthesized to find derivatives of inverse trigonometric functions.

1 If $f(x)=\sin x$, find $f^{-1}(x)$.

Answer: $f(x) = \sin x$, $y = \sin x$, $x = \sin y$, $\arcsin x = y$, $f^{-1}(x) = \arcsin x$

2 For any composition of functions what is $f(f^{-1}(x))$?

Answer: $f(f^{-1}(x)) = x$

3 If $y = \arcsin x$, rewrite x in terms of y.

Answer: If $f^{-1}(x) = y$ then $f(f^{-1}(x)) = f(y)$, therefore as $f(f^{-1}(x)) = x$ so $x = f(y)$

4, 5 Guided steps.

6 Using a trigonometric identity show that cos y can be written as $\sqrt{1-x^2}$

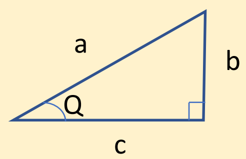
Answer: If $f(y) = x$, $\frac{\mathrm{d}(f(y))}{\mathrm{d}x} = 1$, $\frac{\mathrm{d}f}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}f}{\mathrm{d}y}}$

7 Therefore write the derivative of arcsin x in terms of x.

Answer:

$\cos(\arcsin x) = ?$

Let $Q = \arcsin x$



a = hyp
b = opp
c = adj

$\sin Q = \frac{O}{H} = \frac{b}{a}$

$\sin(\arcsin x) = x$

$\frac{b}{a} = \frac{x}{1}$

$\cos Q = \frac{a}{h} = \frac{\sqrt{1-x^2}}{1}$

$\cos(\arcsin x) = \sqrt{1-x^2}$

8 **Conceptual:** Could this logical reasoning work for arccos x and arctan x?

Answer: Yes, the same reasoning could be extended to arccos$x$ and arctan$x$.

9 Complete this procedure for arccos and arctan.

Answer:

$y = \arccos x$

$x = \cos y$

$\frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y}$

$= -\frac{1}{\sin(\arccos x)}$

$= -\frac{1}{\sqrt{1-x^2}}$

Let $Q = \arccos x$

$\cos Q = \frac{c}{a}$

$\cos(\arccos x) = x$

$\sin(\arccos x) = 1 - x^2$

$y = \arctan x$

$x = \tan y$

Let $Q = \arctan x$

$$\frac{dx}{dy} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{\sec^2}(\arctan x)$$

$$\tan Q = \frac{b}{c} = \frac{x}{1}$$

$$\sec^2 Q = 1 + \tan^2 Q$$

$$\sec^2(\arctan) = 1 + x^2$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

10 Conceptual: How can we use the concepts of inverse and composite functions to find the derivative of an inverse trigonometric function?

Answer: Knowledge of inverse and composite functions along with knowledge of basic trigonometry can be synthesized to find derivatives of inverse trigonometric functions.

TOK

You might want to consider the use of simplicity over accuracy and the value of radians in calculus and physics.

Consider the arbitrary nature of degree measure versus radians as real number and the implications of using these two measures on the shape of sinusoidal graphs.

If we can use different measures to represent the same thing, is mathematics such an international language?

The Sound of Mathematics!

Approaches to Learning: Research, Critical Thinking, Using technology

Exploration Criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB Topic: Trigonometric functions

Introduction

When choosing an exploration topic, students are often encouraged to look for links between one of their interests and mathematics. A topic which is frequently considered by students and not always done well is music and mathematics. Students' explorations tend to be based mainly on research pieces without any real **personal engagement** beyond personal interest, which is not enough to reach the higher levels of Criterion C.

In this task students are encouraged to start by considering the usefulness of a mindmap in producing ideas for ideas to explore within the field of music and then to consider one of their ideas to brainstorm further.

One of the aims of this task is to help students to appreciate that sound simply consists of travelling waves. Students will consider the relationship between pitch, frequency, and period. However, a more powerful aspect of the task is also the opportunity for students to consider where what they know could lead to beyond pure research. Students are asked to think of ideas of how they can explore how trigonometric functions are related to different sounds and musical notes in various different situations. Students also consider what technology it may be appropriate to use.

Brainstorm

To remind students of what a mindmap is, look at the mindmap on Food and Drink in the Exploration chapter of the Student Book on page 742.

This is an opportunity to discuss mindmapping if this has not already been done in the course. Students will probably be familiar with mindmaps from other subjects too.

Split students into groups of 3 or 4 and provide students with large pieces of paper or a whiteboard and give them time to brainstorm and construct mindmaps.

The whole class can then share their mindmaps and discuss.

Research

When a sound is made, say on a musical instrument, it produces **vibrations** in the air. These vibrations vary depending on the sound. Particles vibrating because of the sound cause other nearby particles to vibrate, and so on. These vibrating particles make up a sound wave. Sound waves travel at different speeds and with variations in pressure through air, liquids and solids. (Sound waves are sometimes called pressure waves).

A sound wave can be modelled using a **sine curve**.

The **frequency** determines the **pitch** of a sound (the greater the frequency, the higher the pitch). Frequency is measured in **Hertz** (Hz) and is the number of full **periods** of the soundwave per second. Waves with different Hertz values each have distinct sounds.

The **amplitude** determines the **loudness** (the greater the amplitude the higher the volume).

Encourage students to look up, discuss or research further any of the words in bold that are unfamiliar to them.

This is a good opportunity for individual research as well as class discussion.

Students now have an opportunity to summarize and use what has been learnt in the chapter.

The **Period** goes from one peak to the next (or from any point to the next matching point).

The **Amplitude** is the height from the center line to the peak (or to the trough). Or you can measure the height from highest to lowest points and divide that by 2.

Frequency is how often something happens per unit of time.

For a sine wave with the basic form $y = a\sin(bt)$:

- a is the amplitude
- b is connected to the period and frequency of the function
- y is the output value of the trigonometric function.

$\text{Period} = \dfrac{2\pi}{b}$ and also $\text{Period} = \dfrac{1}{\text{Frequency}}$

A sound with frequency of 440Hz, for example, means there are 440 periods per second (i.e. 1 period per $\frac{1}{440}$ second).

In this case *y* is the pressure.

For a sound of 440 Hz:

- period = $2\pi/b$
- $1/440 = 2\pi/b$
- $b = 880\pi$

So a sound with frequency 440 Hz is modelled by $y = a\sin(880\pi t)$.

For **extension**, students could also investigate the sine wave equations for other musical notes.

Technology

Here are some examples of programmes that can be used to consider sound waves:

Audacity

'Audacity' is a free program that can be downloaded and is designed for sound analysis and editing. Using this program it is possible to record or generate a sound and view a graphical representation of its sound wave.

Note: the measurement setting at the bottom needs to be set to 'Length' and the units to hh:mm:ss - milliseconds. The record bottom will record sounds and the generate tab will produce sounds from the computer to analyse. Students will need to zoom in on the display. The mouse can be used to highlight repetitions of the period and the amount of time taken to complete these periods. This divided by the number of periods will give the actual period.

CBL (Calculator-Based Laboratory system) with connected microphone linked to a TI graphing calculator

This can collect sound data and store it in a graphing calculator. The calculator can then graph the sound as a function of time.

Wolfram Alpha

Wolfram Alpha can be used to play notes with different frequencies.

(There are many other similar free and paid programmes and packages available. You could check which programmes are available in your school's music department, if there is one).

Design an investigation

Here are some possible ideas. This is by no means exhaustive!

- Students could use the generate function in Audacity to create tones with different frequencies and graph and compare the corresponding sine curve. They can also generate tones with the same frequency, but different amplitudes.
- Students could record a steady tone produced by their own voices. They should try to make sure they keep the pitch steady and record the sound for long enough (say 15 seconds). They could then use Audacity or the CBL software to determine whether the note is pitch perfect.

- Students could play notes on an instrument and measure how long it takes for the sound wave to complete, for example, ten periods. From this information they could determine the period and the frequency of this sound and compare the curves produced for different notes - perhaps looking at the relationship between the different notes in a full scale.
- Students could try to generate different notes by adjusting the level of water in a bottle and blowing across the top of it. By altering the amount of water in the bottle can they produce a note of a particular frequency? Can they use this to produce a musical instrument?
- Bring in different musical instruments and play the same pure note on each. Observe the differences between the sine functions when the instrument and volume of the note changes.
- Students could use a programme like Wolfram Alpha to find what frequencies are and are not audible to the human ear. They could design a hearing test, for example, by comparing the different pitches people can hear. This study could be related to age for example.
- Students could consider more complicated sounds. They could sum together several sine waves and see what sounds are produced. This could lead to research into the meaning of consonance and dissonance. Students could try to make and record sounds that are not periodic.

The intention is not necessarily that students do the full exploration here (this may be too time-consuming). This of course would be good. Students may even want to use the ideas as their actual exploration. The idea is that students have engaged in the planning of the exploration and this should be useful for the actual IA.

Extension

Students could consider how they could collect readings such as tide times, sunrise or sunset times, the height of a spring or pendulum, temperature over the course of a day/year, etc.

Having found the data and plotted it students could then try to find the model that best fits the data. This may be in the form $y = a \sin(bt)$, although some more complicated data may require looking for a function of the form $y = a \sin(bt + c) + d$, with different phase shifts and starting points to consider too.

7 Generalizing relationships: exponents, logarithms and integration

In this chapter students will be introduced to integration as anti-differentiations followed by areas under curves as Riemann sums. This leads to the Fundamental Theorem of Calculus. An introduction to exponential and logarithmic rules and exponential and logarithmic functions follows. This allows for further development of calculus involving such functions. The final part of the chapter allows students to understand how to use and apply different techniques of integration, including integration by substitution, by parts as well as cyclic integration by parts. By working through this chapter students would be able to use differential and integral calculus any combination of functions covered in Mathematics: analysis and approaches HL.

Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model relationships between variables, visually or symbolically as graphs, equations and/or tables represents different ways to communicate mathematical ideas.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Logarithm laws provide the means to find inverses of exponential functions which model real-life situations.
- Areas under curves can be approximated by the sum of the areas of rectangles which may be calculated even more accurately using integration.
- Numerical integration can be used to approximate areas in the physical world.
- Mathematical modelling can provide effective solutions to real-life problems in optimization by maximizing or minimizing a quantity, such as cost or profit.
- Equivalent representations of exponential / logarithmic functions can reveal different characteristics of the same relationship.
- Patterns can be identified in behaviours which can give us insight into appropriate strategies to model or solve problems
- Derivatives and integrals describe real-world kinematics problems in two and three-dimensional space by examining displacement, velocity and acceleration.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
The derivative function may provide critical information about the original function such as concavity and increasing or decreasing intervals, however only a family of possible functions can be produced by looking at the derivative function only.	Investigation 1
The integral of a polynomial function is the sum of the integrals of the individual terms of the polynomial.	Investigation 2
The anti-derivative/ integration represents the inverse process of the derivative and as a consequence the inverse of all differentiation rules may be utilized.	Investigation 3
The area bounded by a parabola and the x-axis can be approximated by the sum of a series of triangles constructed within this area.	Investigation 4

Areas under curves can be approximated by the limiting sum of areas of rectangles.	Investigation 5
Areas under curves can be approximated by the sum of rectangles. The limit of the sum, as the number of rectangles tends to infinity, is equal to the definite integral of the curve between the upper and lower bounds.	Investigation 6
The area of a function that is below the x-axis in the interval [a, b] can be found by taking the absolute value of the integral.	Investigation 7
Knowing the basic axioms of mathematics results in drawing the correct conclusions and avoiding ambiguous results.	Investigation 8
The rules of exponents allow for simplification when applying the operations of multiplication and division to very big and/ or very small numbers.	Investigation 9
Logarithms and exponents represent the same relationship in different ways.	Investigation 10
Exponential growth is bigger and faster than polynomial growth of any degree.	Investigation 12
Logarithms represent inverse functions of exponential functions, and vice versa.	Investigation 13
Because logarithms and exponents represent the same relationship in different ways, switching to exponential form enables you to find the derivative of a logarithmic function.	Investigation 14
An understanding that certain questions require application of knowledge from different areas of mathematics will aid solving some more complicated problems.	Investigation 15
Antidifferentiation and the inverse chain rule help to find integrals of composite exponential functions and reciprocal functions.	Investigation 16
Manipulating an integrand to be in a certain form allows you find an appropriate substitution and integrate the expression.	Investigation 17
The cyclical nature of derivatives of $\sin x$ and $\cos x$, and the invariance of the exponential function under differentiation and integration allows us to integrate the product of an exponential function with a trigonometric function.	Investigation 19

Syllabus sections covered in this chapter:

- SL1.5*
- SL1.7
- SL2.9
- SL2.10
- SL2.11
- SL5.6
- SL5.5*
- SL5.10
- SL5.11
- AHL5.15
- AHL5.16

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	*Animated worked example*	*GDC skills and support*	*Additional exercises*
Page 443: Generalizing relationships: exponents, logarithms and integration	Page 462: Example 12 Page 486: Example 33 Page 498: Example 40 Page 501: Example 43	Page 455: Example 6 Page 458: Example 8	Pages 459, 481, 487, 509

Assessment opportunities

End of chapter test	*Chapter review*	*Exam practice*
Page 510	Page 514	Page 514

7.1 Integration as antidifferentiation and definite integrals

Investigation 1

Conceptual understanding:

The derivative function may provide critical information about the original function such as concavity and increasing or decreasing intervals, however only a family of possible functions can be produced by looking at the derivative function only.

1 Over what intervals is $f(x)$ an increasing function?

Answer: $-1 < x < 1$.

2 Over what intervals is $f(x)$ a decreasing function?

Answer: $-2 < x < -1$, $1 < x < 2$

3 Describe the nature of each of the points A to G on the graph of $f(x)$. Explain your answers.

Answer: At A and C there are local maxima as the derivative function moves from being positive to negative.

At B and D there are local minima as the function moves from being negative to positive.

At E, F and G there are points of inflection as the derivative function has turning points, indicating $f''(x)$ changing sign.

4 Over which intervals is $f(x)$ concave up?

Answer: Between E and F, and between G and D, $f(x)$ is concave up since f' is increasing, meaning that $f'' > 0$.

5 Over which intervals is $f(x)$ concave down?

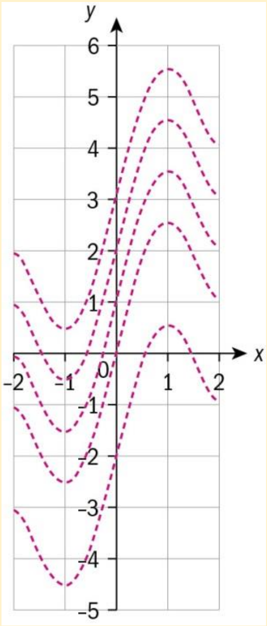
Answer: Between A and E, and between F and G, $f(x)$ is concave down since f' is decreasing, meaning that $f'' < 0$.

6 How accurately can you sketch $y = f(x)$? Give reasons for your answer.

Answer: You can sketch the shape of $f(x)$ accurately, but you cannot tell which points it passes through; i.e. significant points cannot be accurately located.

7 How many possible correct sketches of $y = f(x)$ can be drawn? What do these curves have in common? Give examples.

Answer: There are infinitely many sketches that can be drawn because further information is required to draw an accurate sketch. The possible sketches all have the same gradient at any given value of x.



Conceptual: To what extent does the derivative function limit our knowledge about the original function

Answer: The derivative function may provide critical information about the original function such as concavity and increasing or decreasing intervals, however only a family of possible functions can be produced by looking at the derivative function only.

Investigation 2

Conceptual understanding:

The integral of a polynomial function is the sum of the integrals of the individual terms of the polynomial.

1 Copy and complete the following table.

$\frac{d}{dx}\left(x^2\right)=2x$	$\int 2x\,dx = x^2+c$	$\int x\,dx=\frac{x^2}{2}+c$
$\frac{d}{dx}(x)=1$	$\int 1\,dx = x+c$	$\int 1\,dx = x+c$
$\frac{d}{dx}\left(x^3\right)=3x^2$	$\int 3x^2dx=x^3+c$	$\int x^2dx=\frac{x^3}{3}+c$
$\frac{d}{dx}\left(x^4\right)=4x^3$	$\int 4x^3dx=x^4+c$	$\int x^3dx=\frac{x^4}{4}+c$
$\frac{d}{dx}\left(x^5\right)=5x^4$	$\int 5x^4dx=x^5+c$	$\int x^4dx=\frac{x^5}{5}+c$
$\frac{d}{dx}\left(x^6\right)=6x^5$	$\int 6x^5dx=x^6+c$	$\int x^5dx=\frac{x^6}{6}+c$
$\frac{d}{dx}\left(x^{\frac{3}{2}}\right)=\frac{3}{2}x^{\frac{1}{2}}$	$\int \frac{3}{2}x^{\frac{1}{2}}dx=x^{\frac{3}{2}}+c$	$\int x^{\frac{1}{2}}dx=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{2}{3}x^{\frac{3}{2}}+c$
$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$	$\int nx^{n-1}dx=x^n+c$	$\int x^ndx=\frac{x^{n+1}}{n+1}+c$
$\frac{d}{dx}\left(5x^4\right)=20x^3$	$\int 20x^3dx=5x^4+c$	$\int 5x^3dx=5\int x^3dx=\frac{5x^4}{4}+c$
$\frac{d}{dx}\left(-x^2\right)=-2x$	$\int -2x\,dx=-x^2+c$	$\int -x\,dx=-\int x\,dx$ $=-\frac{1}{2}x^2+c$
$\frac{d}{dx}\left(-\frac{x^4}{6}\right)=-\frac{2x^3}{3}$	$\int -\frac{2}{3}x^3dx=-\frac{1}{6}x^4+c$	$\int -\frac{2}{3}x^3dx=-\frac{2}{3}\int x^3dx$ $=-\frac{1}{6}x^4+c$
$\frac{d}{dx}\left(\frac{kx^{n+1}}{n+1}\right)=kx^n$	$\int kx^ndx=\frac{kx^{n+1}}{n+1}+c$ *Where* $k\in\mathbb{R}$	$\int kx^ndx=k\int x^ndx$ $=\frac{kx^{n+1}}{n+1}+c$

$\frac{d}{dx}(2x^3 - x^2 + 4x)$ $= 6x^2 - 2x + 4$	$\int(6x^2 - 2x + 4)\,dx$ $= 2x^3 - x^2 + 4x + c$	$\int(6x^2 - 2x + 4)\,dx$ $= \int 6x^2 dx + \int -2x dx + \int 4dx$ $= 2x^3 - x^2 + 4x + c$

2 Explain in words how you would find the integral of x^n.

Answer: Increase the exponent by 1 and divide by the new exponent, then add a constant of integration c.

3 What is the rule for integrating kx^n, $k \in \mathbb{R}$?

Answer: The integral of $kx^n, k \in \mathbb{R}$ is the same as $k\int x^n dx$

4 **Conceptual:** What can you say about the integral of a polynomial function?

Answer: The integral of a polynomial function is the sum of the integrals of the individual terms of the polynomial. i.e.

$$\int(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0)dx = \int a_n x^n dx + \int a_{n-1}x^{n-1}dx + \ldots + \int a_1 x dx + \int a_0 dx$$

Investigation 3

Conceptual understanding:

The anti-derivative/ integration represents the inverse process of the derivative and as a consequence the inverse of all differentiation rules may be utilized.

1 a Find the integral of the function by integrating term by term.

Answer: $\int(ax + b)\,dx = \frac{ax^2}{2} + bx + c$

b Expand and simplify $\frac{y^2}{2a}$.

Answer: $\frac{(ax+b)^2}{2a} = \frac{a^2x^2 + 2abx + b^2}{2a} = \frac{ax^2}{2} + bx + \frac{b^2}{2a}$

c How can you combine your results to parts **a** and **b** in order to integrate $y=ax+b$ without integrating each term separately?

Answer: $\int(ax + b)\,dx = \frac{(ax+b)^2}{2a} + c$

2 a Expand y^2 and hence find $\int(ax+b)^2\,dx$.

Answer: a $y^2 = a^2x^2 + 2abx + b^2 \Rightarrow \int(ax+b)^2\,dx = \frac{a^2x^3}{3} + abx^2 + b^2x + c$

b Expand and simplify $\frac{y^3}{3a}$.

Answer: $\frac{y^3}{3a} = \frac{a^3x^3 + 3a^2x^2b + 3axb^2 + b^3}{3a} = \frac{a^2x^3}{3} + ax^2b + xb^2 + \frac{b^3}{3a}$

c How can you combine your results to parts a and b in order to integrate y^2 without integrating each term separately?

Answer: $\int (ax+b)^2 \, dx = \frac{(ax+b)^3}{3a} + c$

3 a Expand y^3 and hence find $\int (ax+b)^3 \, dx$.

Answer: $y^3 = a^3x^3 + 3a^2x^2b + 3axb^2 + b^3$

$$\int y^3 dx = \frac{a^3x^4}{4} + a^2x^3b + \frac{3ax^2b^2}{2} + b^3x + c$$

b Expand and simplify $\frac{y^4}{4a}$.

Answer: $\frac{y^4}{4a} = \frac{a^4x^4 + 4a^3x^3b + 6a^2x^2b^2 + 4axb^3 + b^4}{4a}$

$$= \frac{a^3x^4}{4} + a^2x^3b + \frac{3ax^2b^2}{2} + b^3x + \frac{b^4}{4a}$$

c How can you combine your results to parts **a** and **b** in order to integrate y^3 without integrating each term separately?

Answer: $\int (ax+b)^3 \, dx = \frac{(ax+b)^4}{4a} + c$

4 Based on your results above, make a conjecture about the value of the integral $\int (ax+b)^n \, dx$.

Answer: $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

5 Find the derivatives of the following functions.

a $\sin(ax+b)$

Answer: $\frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$

b $\cos(ax+b)$

Answer: $\frac{d}{dx}(\cos(ax+b)) = -a\sin(ax+b)$

c $\tan(ax+b)$

Answer: $\frac{d}{dx}(\tan(ax+b)) = a\sec^2(ax+b)$

6 Use your results to question **5** to find the following integrals.

a $\int \cos(ax+b) dx$

Answer: $\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + c$

b $\int \sin(ax+b)\mathrm{d}x$

Answer: $\int \sin(ax+b)\,\mathrm{d}x = -\dfrac{\cos(ax+b)}{a}+c$

c $\int \sec^2(ax+b)\mathrm{d}x$

Answer: $\int \sec^2(ax+b)\mathrm{d}x = \dfrac{\tan(ax+b)}{a}+c$

7 **Factual:** What differentiation rule have you used the reverse of in this investigation?

Answer: The inverse chain rule.

8 **Conceptual:** Why is knowledge of derivatives useful for integration?

Answer: The anti-derivative / integration represents the inverse process of the derivative and as a consequence the inverse of all differentiation rules may be utilized.

Investigation 4

Conceptual understanding:

The area bounded by a parabola and the *x*-axis can be approximated by the sum of a series of triangles constructed within this area.

1 Find the area of triangle ABC.

Answer: Area of triangle ABC $=\dfrac{1}{2}\times 4\times 4=8$ units²

2 Write an inequality relating the area bounded by the parabola and the x-axis and the area of the triangle.

Answer: Area under parabola > 8 units²

3 Show that the area of each of the triangles ADC and CEB is equal to 1 unit².

Answer: Triangles ADC and CEB are congruent through symmetry about the *y*-axis so you only need to find area of 1 triangle, e.g. triangle ADC.

The area is found easily by dropping a perpendicular from D to the x-axis, and then finding the difference of two right angled triangles and two trapezia giving:

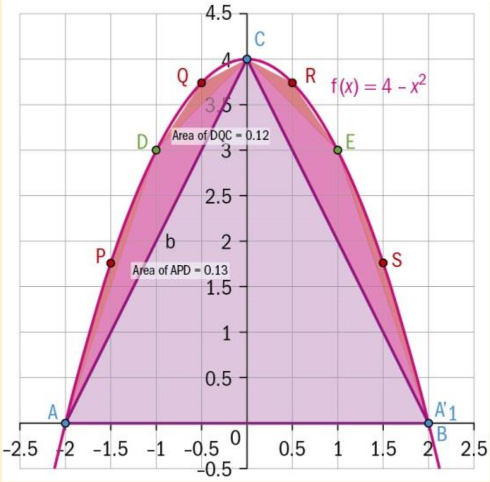
Area of ADC$=\left\lfloor \dfrac{1}{2}(1\times 3)-\dfrac{1}{2}(1\times 2)\right\rfloor+\left\lfloor \dfrac{1}{2}(3+4)\times 1-\dfrac{1}{2}(2\times 4)\times 1\right\rfloor=1$ unit²

4 Extend your inequality from question **2** to relate the area under the curve to the sum of the areas under all three triangles.

Answer: Area under Parabola > 10 units²

5 Construct points P, Q, R and S on the parabola which have x-coordinates at −1.5, −0.5, 0.5, 1.5 respectively.

Answer: The diagram shows the parabola with the points and triangles constructed.



6 Calculate the areas of triangles APD, DQC, CRE and ESB.

Answer: Area of triangle APD = Area of triangle ESB = 0.13 units²

Area of triangle DQX = Area of triangle CRE = 0.12 units²

7 Extend your inequality from question **2** to relate the area under the curve to the sum of the areas under all the triangles.

Answer: Area under parabola > 10 .50 units²

8 If you were to continue drawing further triangles in this way, what could you say about the sum of their areas?

Answer: You can continue constructing triangles in this way and each time sum of the areas of the triangles is a better approximation to the area under the parabola.

9 **Conceptual:** How does the method of exhaustion allow you to find a good approximation of the area bounded by the parabola and the *x*-axis?

Answer: The area bounded by a parabola and the *x*-axis can be approximated by the sum of a series of triangles constructed within this area.

Investigation 5

Conceptual understanding:

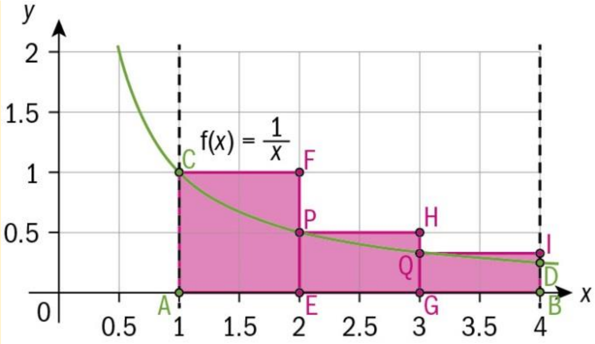
Areas under curves can be approximated by the limiting sum of areas rectangles.

1 & 2 Copy the diagram and mark on points P and Q.

On your diagram, construct three rectangles as follows.

- Rectangle AEFC.E is the point (2,0) and FC is a line segment parallel to the *x*-axis.
- Rectangle EGHP. G is the point (3,0) and PH is a line segment parallel to the *x*-axis.
- Rectangle GBIQ. IQ is a line segment parallel to the *x*-axis.

Answer:



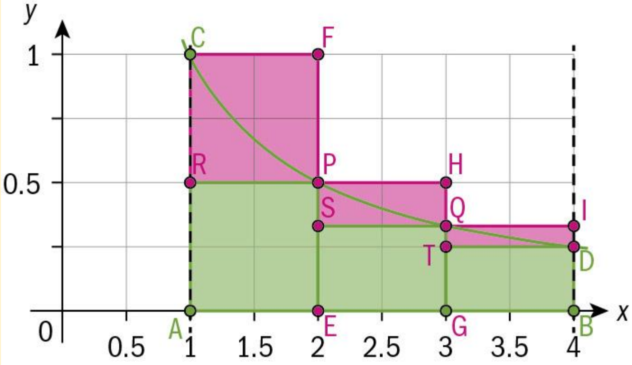
3 Calculate the area of each of the three rectangles.

Answer: $\left.\begin{aligned} AEFC &= 1\times 1 \\ EGHP &= 1\times\frac{1}{2} \\ GBIQ &= 1\times\frac{1}{3}\end{aligned}\right\}$ sum of areas $=\frac{11}{6}\approx 1.833$

4 On the same diagram, construct three more rectangles as follows.

- Rectangle AEPR. PR is a line segment parallel to the *x*-axis.
- Rectangle EGQS. QS is a line segment parallel to the *x*-axis.
- Rectangle GBDT. DT is a line segment parallel to the *x*-axis.

Answer:



5 Calculate the area of each of these three rectangles.

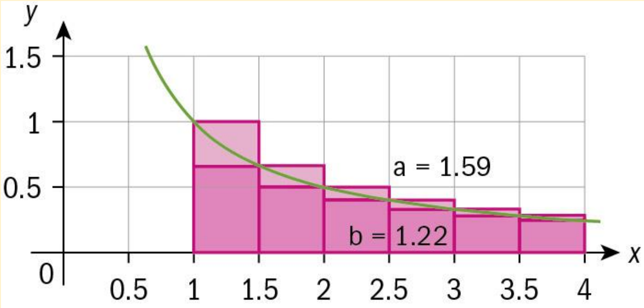
Answer: Area of rectangles: $\left.\begin{aligned} AEPR &= 1\times\frac{1}{2} \\ EGQS &= 1\times\frac{1}{3} \\ GBDT &= 1\times\frac{1}{4}\end{aligned}\right\}$ Sum of areas $=\frac{13}{12}\approx 1.083$

6 Write an inequality relating both sums of the areas of rectangles (from question **3** and question **5**) to the area ACDB.

Answer: $1.083 < \text{area ACDB} < 1.833$

7 Repeat method described in questions 2 to 6, but this time construct a set of six rectangles which lie below the curve, and another set of six rectangles which lie above the curve. (Hint: The bases of each of the six rectangles should be determined by the points (1, 0), (1.5, 0), (2, 0), (2.5, 0), (3, 0), (3.5, 0), (4, 0).)

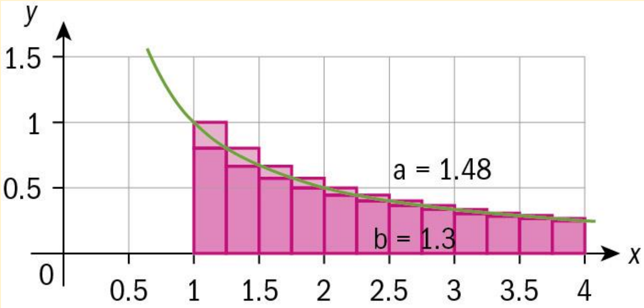
Answer: Students may use various types of technology for this section. The answers and diagrams below were obtained using Geogebra.



$1.22 < \text{Area } ACDB < 1.59$

8 What happens if you were to repeat the above using two sets of 12 rectangles?

Answer:



$1.3 < \text{Area ACDB} < 1.48$

9 Write an inequality relating the areas of two sets of n rectangles, each with base width Δx, with the area ACDB.

Answer: $\displaystyle\sum_{i=1}^{n} \frac{1}{(x_i + \Delta x)} \times \Delta x < \text{Area ACDB} < \sum_{i=1}^{n} \frac{1}{x_i} \times \Delta x$

10 Conceptual: How does the concept of limits lead to an approximate measure of area under a curve?

Answer: Areas under curves can be approximated by the limiting sum of areas rectangles.

$$\text{Area under curve} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{x_i} \times \Delta x$$

Investigation 6

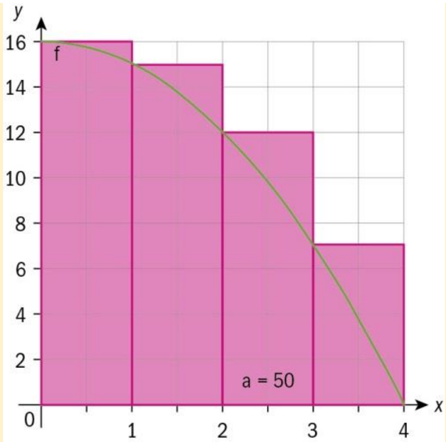
Conceptual understanding:

Areas under curves can be approximated by the sum of rectangles. The limit of the sum, as the number of rectangles tends to infinity, is equal to the definite integral of the curve between the upper and lower bounds.

1 & 2 Draw the graph of the function $f(x) = 16 - x^2$ for $0 \le x \le 4$.

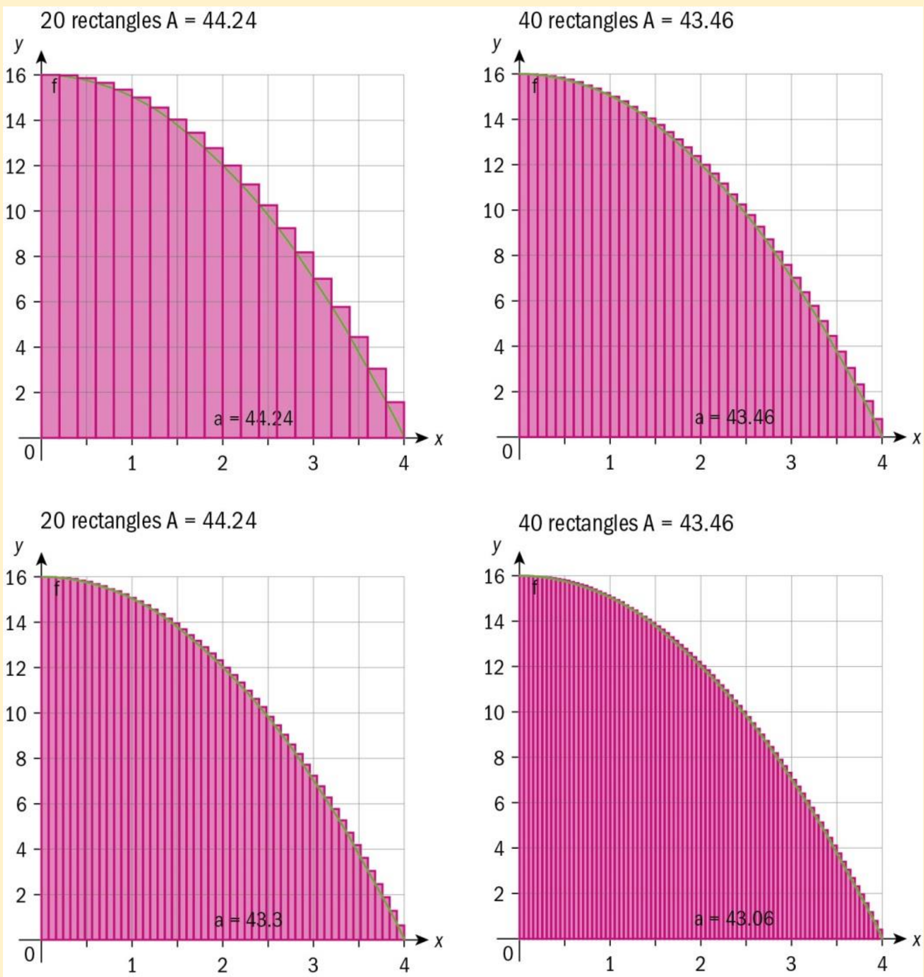
Answer: Area =50 units²

Draw a Riemann upper sum with four rectangles and find this sum.



3 Use technology to increase the number of rectangles to 20, 40, 50 and 80. Calculate the respective Riemann upper sums.

Answer:



4 Find the value of the indefinite integral $\int\left(16-x^2\right)\mathrm{d}x$.

Answer: $\int\left(16-x^2\right)\mathrm{d}x = 16\int 1\mathrm{d}x - \int x^2\mathrm{d}x \ = 16x - \dfrac{x^3}{3} + c$

5 **a** Substitute $x = 4$ in your answer.

Answer: When $x = 4, 16x - \frac{x^3}{3} + c = 64 - \frac{64}{3} + c \simeq 42.67 + c$

b Now substitute $x = 0$ in your answer.

Answer: When $x = 0, 16x - \frac{x^3}{3} + c = c$

c Calculate the difference of these two answers.

Answer: 42.67

6 Repeat questions **1** to **5** for the following functions.

a $f(x) = 8 - x^3$, $0 \le x \le 2$. Substitute $x = 2$ and $x = 0$ into the indefinite integral, and find the difference.

Answer: Upper Riemann sums for A : 13.75, 12.39, 12.2, 12.16, 12.2

$$\int (8 - x^3)\,dx = 8\int 1\,dx - \int x^3\,dx \;= 8x - \frac{x^4}{4} + c$$

$$\text{When } x = 2, 8x - \frac{x^4}{4} + c = 16 - 4 + c = 12 + c$$

$$\text{When } x = 0, 8x - \frac{x^4}{4} + c = c$$

Difference = 12

b $f(x) = \cos x$, $0 \le x \le \frac{\pi}{2}$. Substitute $x = \frac{\pi}{2}$ and $x = 0$ into the indefinite integral, and find the difference.

Answer: Upper Riemann sums for A: 5.92, 5.19, 5.1, 5.08, 5.05

$$\int 5\cos x\,dx = 5\int \cos x\,dx \;= 5\sin x + c$$

$$\text{When } x = \frac{\pi}{2}, 5\sin x + c = 5\sin\frac{\pi}{2} + c = 5 + c$$

$$\text{When } x = 0, 5\sin x + c = 5\sin 0 + c = c$$

Difference = 5

7 What do you notice about your results?

Answer: The Riemann sums were all approximately equal to the difference of the integrals.

All were bigger than the difference.

As the number of rectangles increased the Upper Riemann sum was closer to the integral difference.

8 How would you expect your answer to question **7** to be different if you had been asked to find Riemann lower sums for each function?

Answer: If lower Riemann sums were taken the answers would also be approximately equal to the integral difference but this time they would all be less than the difference.

9 **Conceptual:** How does your understanding of limits lead to an accurate measure of area under a curve?

Answer: Areas under curves can be approximated by the sum of rectangles. The limit of the sum, as the number of rectangles tends to infinity, is equal to the definite integral of the curve between the upper and lower bounds.

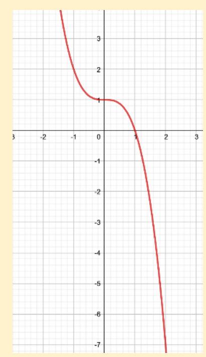
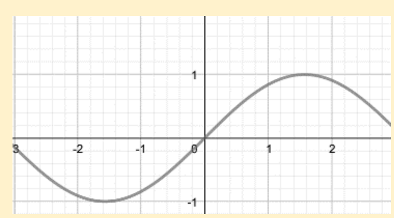
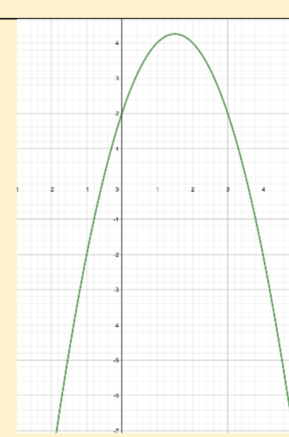
Investigation 7

Conceptual understanding:

The area of a function that is below the *x*-axis in the interval [*a*, *b*] can be found by taking the absolute value of the integral.

1 Copy and complete the following table.

Answer:

Definite integral $\int_a^b f(x)\,dx$	***Numerical answer***	***Sketch of*** $f(x)$
$\int_{-2}^{2}(1-x^3)\,dx$	4	
$\int_{-2}^{1}(1-x^3)\,dx$	6.75	
$\int_{1}^{2}(1-x^3)\,dx$	−2.75	
$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}\sin x\,dx$	0.707	
$\int_{-\frac{\pi}{4}}^{0}\sin x\,dx$	-0.293	
$\int_{0}^{\frac{\pi}{2}}\sin x\,dx$	1	
$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\,dx$	0	
$\int_{-2}^{2}(2+3x-x^2)\,dx$	2.67	
$\int_{-2}^{0}(2+3x-x^2)\,dx$	−4.67	
$\int_{0}^{2}(2+3x-x^2)\,dx$	2.67	

2 **Factual:** Why do some of the answers to question 1 not give the total area bounded by the graph, the x-axis and the upper and lower limits?

Answer: The numerical value of the integral for the areas under the x-axis are all negative. If, between the limits of integration, part of the graph is above the x-axis and part below, then the positive and negative values will cancel each other out.

3 **Conceptual:** Explain how you could you use integration to calculate the area between a function and the x-axis when the function is fully below the x-axis.

Answer is the TU: The area of a function that is below the x-axis in the interval $[a, b]$ can be found by taking the absolute value of the integral.

TOK

A useful starting point for many TOK questions is to identify the key themes and words in the knowledge question and look for examples, claims and counterclaims.

- What do we mean by a problem?
- What instances do you know of where mathematics has been created to solve a problem?
- Can you have knowledge that does not solve problems?

Developing inquiry skills

In the opening scenario for this chapter, the function $g(t) = \dfrac{1}{50}e^{-\frac{t}{10}+3.09}$ models the rate of growth of the hydrangea bush in meters per day for $t \geq 20$ days. Find the antiderivatives of the function g.

Answer: $-\dfrac{1}{5}e^{-\frac{t}{10}+3.09} + C$

Investigation 8

Conceptual understanding:

Knowing the basic axioms of mathematics results in drawing the correct conclusions and avoiding ambiguous results.

1 Use the definition that $a^m = \underbrace{a \times a \times \ldots \times a}_{m}$ to show the following properties are true.

a $a^m \times a^n = a^{m+n}$

Answer: Using the definition

LHS $= \underbrace{a \times a \times \ldots \times a}_{m} \times \underbrace{a \times a \times \ldots \times a}_{n} = \underbrace{a \times a \times \ldots \times a}_{m+n} = a^{m+n}$ = RHS

b $a^m \div a^n = a^{m-n}$

Answer: Using the definition

LHS$= \frac{a^m}{a^n} = \frac{\overbrace{a\times a\times\ldots\times a}^{m}}{\underbrace{a\times a\times\ldots\times a}_{n}} \times = \underbrace{a\times a\times\ldots\times a}_{m-n} = a^{m-n}$ = RHS since $m,n \in \mathbb{Z}^+, m \geq n$

c $\left(a^m\right)^n = a^{mn}$

Answer: $(a^m)^n = \left(\underbrace{a\times a\times\ldots\times a}_{m}\right)^n$

$$= \left(\underbrace{a\times a\times\ldots\times a}_{m}\right)\times\left(\underbrace{a\times a\times\ldots\times a}_{m}\right)\times\ldots\times\left(\underbrace{a\times a\times\ldots\times a}_{m}\right)$$

$$= \underbrace{a\times a\times\ldots\times a}_{mn} = a^{mn}$$

2 Use any combination of the three properties in question 1 to show that the following are true.

a $a^0 = 1$

Answer: When $m = n$ property 1**b** gives $1 = \frac{a^m}{a^m} = a^{m-m} = a^0$

b $a^{-n} = \frac{1}{a^n}, a \neq 0$.

Answer: Using property **1b** and the answer to **2a** gives $\frac{1}{a^n} = \frac{a^0}{a^n} = a^{0-n} = a^{-n}$

c $a^{\frac{1}{m}} = \sqrt[m]{a}, a > 0$

Answer:

$$a^1 = a^{\overbrace{\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}}^{n}}$$

$$= \underbrace{a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \times a^{\frac{1}{n}}}_{n} \quad \text{by property 1a}$$

$$= \left(a^{\frac{1}{n}}\right)^n$$

Which by the property of roots $\Rightarrow a^{\frac{1}{n}} = \sqrt[n]{a}$

d $a^{\frac{m}{n}} = \sqrt[n]{a^m}, a > 0$

Answer:

$$a^{\frac{m}{n}} = a^{m\times\frac{1}{n}}$$

$$= \left(a^m\right)^{\frac{1}{n}} \quad \text{by property 1c}$$

$$= \sqrt[n]{a^m} \quad \text{by the result in 2c}$$

However, it is also true that

$$a^{\frac{m}{n}} = a^{\frac{1}{n}\times m}$$

$$= \left(a^{\frac{1}{n}}\right)^m \quad \text{by property 1c}$$

$$= \left(\sqrt[n]{a}\right)^m \quad \text{by result in 2c}$$

Hence $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m, a > 0$

3 Explain why $a \neq 0$ is a necessary condition for property **2b** to be true.

Answer: In property **2b** $a \neq 0$ because division by 0 is undefined.

4 Give an example to show that if $a \leq 0$, inconsistencies may arise for property **2c**.

Answer: If we consider $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$ which is always true.

However, $(-27)^{\frac{1}{3}} = (-27)^{\frac{2}{6}} = \sqrt[6]{(-27)^2} = \sqrt[6]{729} = \pm 3$ giving +3 also as a possible answer to $\sqrt[3]{-27}$.

5 Explain why $a^x > 0$ for all $a, x \in \mathbb{R}, a>0$.

Answer: First let us consider $x > 0$ Then $a^x > 0$ when $a > 0$ and $x > 0$

When $x = 0$, $a^x = a^0 = 1 > 0$

When $x < 0$ let $x = -y, y > 0$

Then $a^x = a^{-y} = \frac{1}{a^y} = \left(\frac{1}{a}\right)^y > 0$ since $a,\ y > 0$

6 **Conceptual:** Why is it important to know that mathematics is axiomatic?

Answer: Knowing the basic axioms of mathematics results in drawing the correct conclusions and avoiding ambiguous results.

Investigation 9

Conceptual understanding:

The rules of exponents allow for simplification when applying the operations of multiplication and division to very big and/or very small numbers.

1 Use table **1** to calculate the following:

a 729 × 27

Answer: $729 \times 27 = 3^6 \times 3^3 = 3^{6+3} = 3^9 = 19\,683$

b 243 ÷ 27

Answer: $243 \div 27 = 3^5 \div 3^3 = 3^{5-3} = 3^2 = 9$

c $\frac{27}{729} - \frac{1}{27}$

Answer:

$$\frac{27}{729} - \frac{1}{27} = \frac{3^3}{3^6} - \frac{3^0}{3^3}$$
$$= 3^{3-6} - 3^{0-3}$$
$$= 3^{-3} - 3^{-3}$$
$$= 0$$

2 Use the table to verify the following statements:

a $\frac{81}{2187} + \frac{486}{6561} = \frac{1}{9}$

Answer:

$$\frac{81}{2187} + \frac{486}{6561} = \frac{3^4}{3^7} + \frac{486}{3^8}$$
$$= \frac{3^5}{3^8} + \frac{486}{3^8}$$ multiplying top and botttom of first fraction by 3
$$= \frac{243 + 486}{3^8}$$
$$= \frac{729}{3^8}$$
$$= \frac{3^6}{3^8} = \frac{1}{9}$$

b $\frac{45}{1458} + \frac{243}{13122} = \frac{1}{81}$

Answer:

$$\frac{45}{1458} - \frac{27}{13122} = \frac{5}{162} - \frac{1}{486}$$
$$= \frac{15}{486} - \frac{1}{486}$$

3 Copy and complete table **2** using powers of 2.

Answer:

n	2^n
-5	$\frac{1}{32}$
-4	$\frac{1}{16}$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$

0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512

4 Students' own answers.

5 **Conceptual:** How has the use of exponents helped in calculations?

Answer: The rules of exponents allow for simplification when applying the operations of multiplication and division to very big and/or very small numbers.

Investigation 10

Conceptual understanding:

Logarithms and exponents represent the same relationship in different ways.

1 Use the definition of logarithms and the properties of exponents to verify the following:

a $\log_a x + \log_a y = \log_a xy$

Answer: Let $\log_a x = m$ and $\log_a y = n$

$\Rightarrow a^m = x \text{ and } a^n = y$

$\Rightarrow xy = a^m \times a^n$

$\Rightarrow xy = a^{m+n}$

Therefore $\log_a xy = m + n = \log_a x + \log_a y$

b $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

Answer: Let $\log_a x = m$ and $\log_a y = n$

$\Rightarrow a^m = x$ and $a^n = y$

$\Rightarrow \frac{x}{y} = a^m \div a^n$

$\Rightarrow \frac{x}{y} = a^{m-n}$

Therefore $\log_a\left(\frac{x}{y}\right) = m - n = \log_a x - \log_a y$

c $\log_a x^n = n\log_a x$

Answer: $\log_a x^n = \log_a(\overbrace{x \times x \times x \times x \times \ldots \times x}^{n \text{ times}}) = \underbrace{\log_a x + \log_a x + \log_a x + \ldots + \log_a x}_{ntimes} = n\log_a x$

2 Show that:

a $\log_a 1 = 0$

Answer: Let $\log_a 1 = p$

$\Rightarrow a^p = 1 \Rightarrow p = 0$

Therefore $\log_a 1 = 0$

b $\log_a a = 1$

Answer: Let $\log_a a = q$

$\Rightarrow a^q = a \Rightarrow q = 1$

Therefore $\log_a a = 1$

c $-\log_a x = \log_a\left(\frac{1}{x}\right)$

Answer: $-\log_a x = 0 - \log_a x = \log_a 1 - \log_a x = \log_a\left(\frac{1}{x}\right)$

3 For which values of *a*, *x* and *y* are these properties valid?

Answer: $a, x, y \in R, x, y > 0$

4 Show that $\log_a x = \frac{\log_b x}{\log_b a}$. , hence show that $\log_a b = \frac{1}{\log_b a}$

Answer: Let $\log_a x = y$

Then $x = a^y$ write in exponent form

$\Rightarrow \log_b x = \log_b a^y$ take logarithms to base b of both sides

$\Rightarrow \log_b x = y\log_b a$ bring the power y down

Using the change of base rule above we obtain $\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$

5 **Conceptual:** What can you deduce about logarithms and exponents from your results to this investigation?

Answer: Logarithms and exponents represent the same relationship in different ways.

Investigation 11

This investigation does not have a conceptual understanding. It is intended to be an introduction to help students discover that the sequence $\left(1+\frac{1}{n}\right)^n$ tends to a limit. The value of this limit is Euler's number, e, which will be introduced after the investigation.

1 If €1000 is invested in a bank that offers an interest rate of 2% compounded annually how much is the investment worth after *m* years?

Answer: $1000(1.02)^m$

2 How much is the investment worth after m years if the interest were compounded:

a every six months?

Answer: $1000\left(1+\frac{0.02}{2}\right)^{\frac{m}{2}}$

b monthly?

Answer: $1000\left(1+\frac{0.02}{12}\right)^{\frac{m}{12}}$

3 For each of the three cases above evaluate the value of the investment after exactly one year.

Answer:

Interest compounded annually:	€1020
Interest compounded every six months:	€1020.10
Interest compounded monthly:	€1020.18

4 Investigate the growth of €1 invested for one year at 100% interest, compounded at n different intervals over the year.

a Write down the general formula to obtain the growth of this investment after one year.

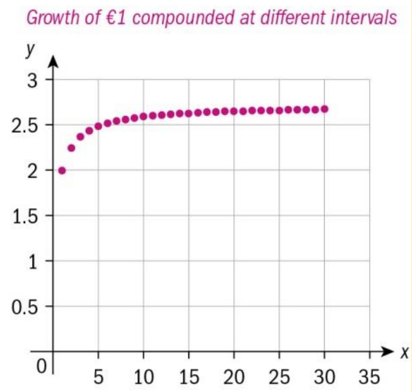
Answer: $1\left(1+\frac{1}{n}\right)^n$

b Use technology to draw up a table with the value of the investment for different values of *n*.

c Plot these values and comment on your results.

Answers:

n	$1\left[1+\frac{1}{n}\right]^{n}$
1	2
2	2.25
3	2.37037
4	2.441406
5	2.48832
6	2.521626
7	2.5465
8	2.565785
9	2.581175
10	2.593742
11	2.604199
12	2.613035
13	2.620601
14	2.627152
15	2.632879
16	2.637928
17	2.642414
18	2.646426
19	2.650034
20	2.653298
21	2.656263
22	2.65897
23	2.66145
24	2.663731
25	2.665836
26	2.667785
27	2.669594
28	2.671278
29	2.672849
30	2.674319



The value of the investment starts to grow quickly for small *n*, but as the value of *n* increases the investment increases much more slowly.

TOK

Some mathematical constants like π , e and the Fibonacci numbers appear consistently in nature. Research where these may be found and consider if they are natural occurrences or are we applying the mathematics that we know to these instances?

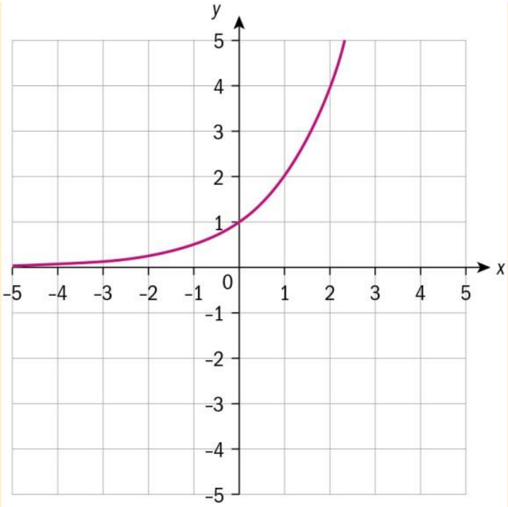
Investigation 12

Conceptual understanding:

Exponential growth is bigger and faster than polynomial growth of any degree.

1 Sketch the function f(x) = 2x, for $-6 \le x \le 6$, $-6 \le y \le 6$.

Answer:



2 **a** Stationary points

Answer: The graph has no stationary points

b Intercepts

Answer: The graph has no x-intercepts and is always positive. The y-intercept is at (0, 1).

c Asymptotes

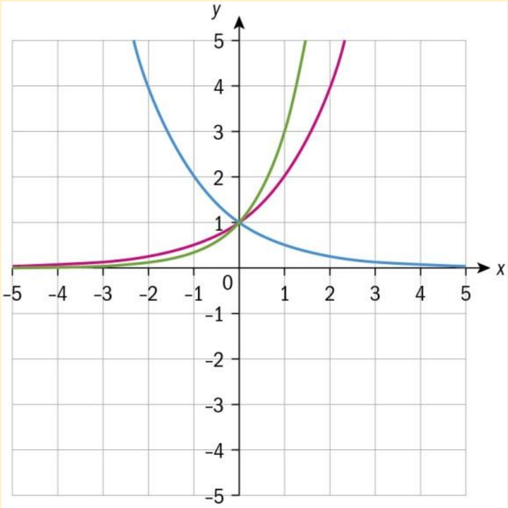
Answer: The x-axis is a horizontal asymptote

d concavity.

Answer: The graph is always increasing, so it is concave up.

3 Using the same scale and axes, sketch the functions $g(x) = a^x$ for different values of $a \in \mathbb{R}^+$.

Answer: Various graphs of this type are possible.



4 What do the graphs have in common?

Answer: All the graphs have a y-intercept at (0, 1) and have the same basic shape.

5 **a** What do the graphs of $g(x) = a^x$, $a > 1$ have in common?

Answer: The graphs of $g(x) = a^x, a > 1$ are all increasing and pass through the point $(0,1)$.

b Why do we refer to these graphs as representations of exponential growth?

Answer: These graphs show exponential growth since, as the value of x increases, the value of the function also increases (i.e. 'grows') according to the 'exponent' x.

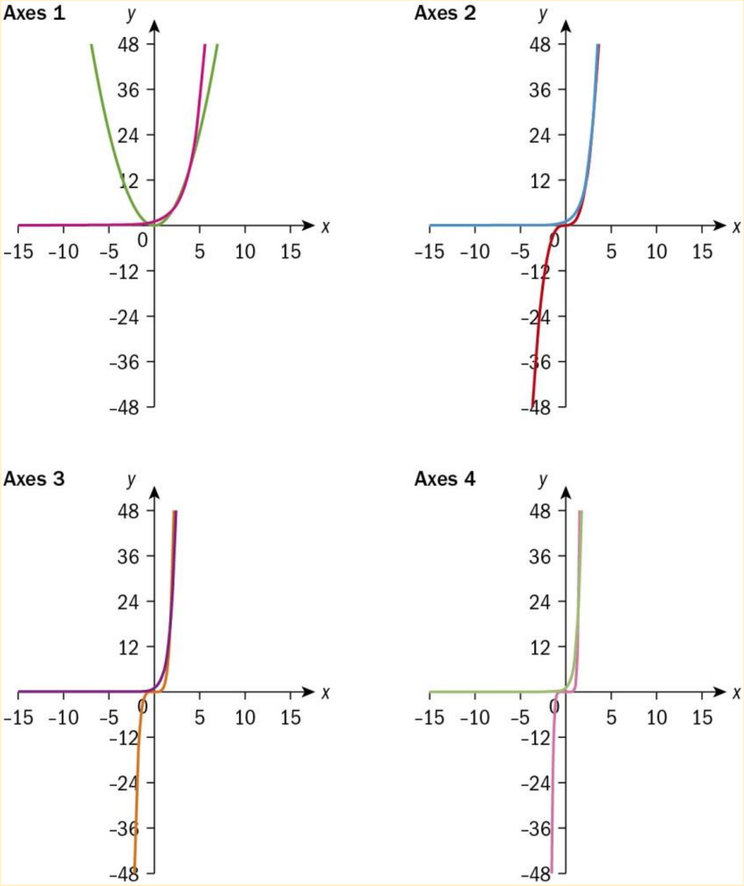
c How are the graphs of $g(x) = a^x$, $0 < a < 1$ different? Why do they represent exponential decay?

d Why do we put the following restrictions on a, $a>0$, $a \neq 1$?

Answer: The graphs of $g(x) = a^x, 0 < a < 1$ are all decreasing.

e **Conceptual:** Compare the graphs of $g(x) = a^x$ and $h(x) = x^a$ for $a \in \{2, 3, 5, 9\}$ to help you answer the question: How does exponential growth compare to polynomial growth?

Answer:



Answer: Exponential growth is bigger and faster than polynomial growth of any degree.

They represent exponential decay since, as the value of x increases, the value of the function decreases (i.e. 'decays') according to the 'exponent' x.

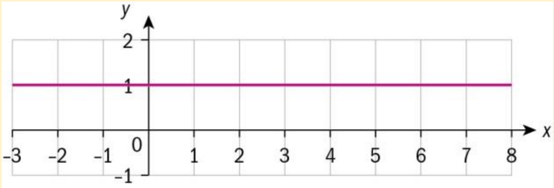
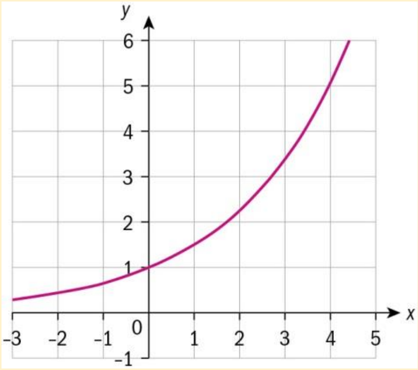
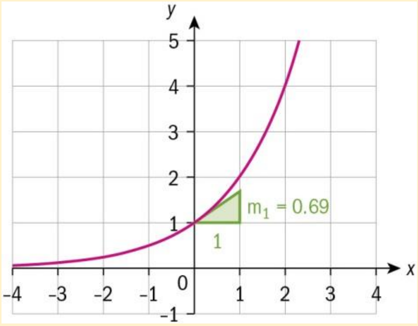
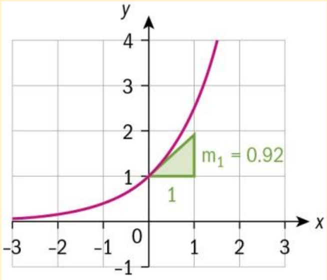
$a = 1$ would be equivalent to the constant function $y = 1$

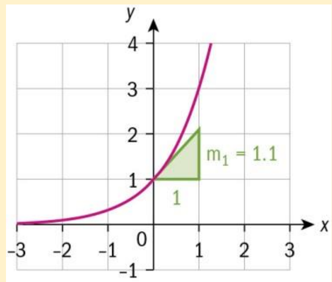
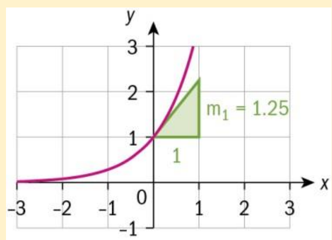
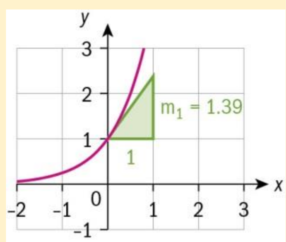
If $a = 0$ this would be equivalent to the x-axis.

The restriction $a > 0$ ensures that we avoid having complex values. e.g.

$$f(x) = (-2)^x \Rightarrow f\left(\frac{1}{2}\right) = (-2)^{\frac{1}{2}} = \sqrt{-2}$$

6 Use technology to copy and complete the table below by graphing the function $g(x) = a^x$ for $a \in \{1, 1.5, 2, 2.5, 3, 3.5, 4\}$ and finding the derivative of each function at $x = 0$.

a	***Graph of*** a^x	$\left.\frac{dy}{dx}\right\|_{x=0}$
1		0
1.5		0.41
2		0.69
2.5		0.92

3		1.1
3.5		1.25
4		1.39

7 **Conceptual:** Conceptual: What happens to the value of $g'(0)$ as a increases?

Answer: The gradient of the function $g(x) = a^x$ at (0, 1) increases as the value of a increases.

TOK

How does exponential growth in mathematics differ from its use in English?

Is language an inadequate vehicle for expressing everything we can experience and think?

Investigation 13

Conceptual understanding:

Logarithms represent inverse functions of exponential functions, and vice versa.

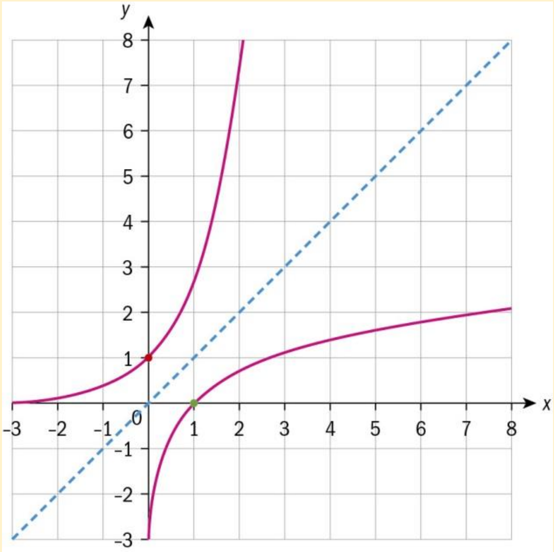
1 **Factual:** Explain why an exponential function $f(x) = a^x, a > 0$ has an inverse.

Answer: The inverse of an exponential function exists because $f(x) = a^x$ is one-to-one, continuous and defined for all values of $x \in \mathbb{R}$.

2 **Factual:** Explain how you would obtain the graph of the inverse of $f(x) = a^x$

Answer: The inverse of a function is given by reflecting the graph of the function in the line $y = x$. As such, the inverse passes through the point (1, 0) and has the y-axis as a vertical asymptote. You can use this information to sketch the graph of the inverse function.

3 On the same axes, sketch the graphs of $f(x) = a^x$ and its inverse.



4 Find the equation of the inverse function $f^{-1}(x)$ of the function $f(x)=a^x$. State the domain and range of the inverse.

Answer: $y = a^x \Rightarrow x = \log_a y$ making x the subject of the formula $\Rightarrow y = \log_a x$

interchanging x and y $\Rightarrow f^{-1}(x) = \log_a x, x, y \in \mathbb{R}, x > 0$

5 **Conceptual:** What do your answers tell you about the relationship between exponential and logarithmic functions?

Answer: Logarithms represent inverse functions of exponential functions, and vice versa.

Investigation 14

Conceptual understanding:

Because logarithms and exponents represent the same relationship in different ways, switching to exponential form enables you to find the derivative of a logarithmic function.

1 Write $y = a^x$ in an equivalent form using a logarithm.

Answer: $y = a^x \Leftrightarrow \log_a y = x$

2 Write $\ln y = x$ in an equivalent form using an exponential.

Answer: $y = e^x \Leftrightarrow \ln y = x$

3 Explain why exponential and logarithmic functions are differentiable over their domain.

Answer: Logarithmic and exponential functions are differentiable over their domain because they are continuous and if you draw a tangent at any point the tangent is never vertical. Also, the functions are smooth and have no abrupt bends or cusps.

4 Use your answer to question **2** to find the derivative of the function $y = \ln x$ by using implicit differentiation.

Answer: $y = \ln x \Rightarrow x = e^y$

Implicit differentiation: $\Rightarrow 1 = e^y \dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{e^y} = \dfrac{1}{x}$

5 Determine the derivative of the function $y = \log a^x$.

Answer: $y = \log_a x \Rightarrow y = \dfrac{\ln x}{\ln a}$ change base from a to e

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln x) = \frac{1}{x\ln a}$$

6 In section **7.2** you used differentiation from first principles to show that $\dfrac{d}{dx}\left(a^x\right) = ka^x$, where $k = \lim\limits_{h\to 0}\dfrac{a^h - 1}{h}$. Use your result from question **4** to find the value of k.

Answer: $y = a^x \Rightarrow \ln y = \ln a^x$

$\Rightarrow \ln y = x\ln a$ using laws of logarithms

$\dfrac{1}{y}\dfrac{dy}{dx} = \ln a$ Differentiating implicitly with respect to x and using the result in question 4

$$\Rightarrow \frac{dy}{dx} = (\ln a)y = (\ln a)a^x$$

Hence $k = \lim\limits_{h\to 0}\dfrac{a^h - 1}{h} = \ln a$

7 What do your results imply about the derivative of:

a exponential functions at the y-intercept

Answer: $\dfrac{d}{dx}\left(a^x\right) = (\ln a)a^x$

$$\Rightarrow \left.\frac{d}{dx}\left(a^x\right)\right|_{x=0} = \ln a$$

b the exponential function $y = e^x$ at the y-intercept?

Answer: $\dfrac{d}{dx}\left(e^x\right) = (\ln e)e^x$

$$\Rightarrow \left.\frac{d}{dx}\left(e^x\right)\right|_{x=0} = \ln e = 1$$

8 **Conceptual:** From what you have learned in this investigation, how is it useful to be able to switch between exponential and logarithmic form?

Answer: Because logarithms and exponents represent the same relationship in different ways, switching to exponential form enables you to find the derivative of a logarithmic function.

TOK

Are logarithms a natural occurrence or are they a human invention?

Is mathematics created to solve real world problems?

Do you need imagination to create new mathematics?

Does faith have a role to play in the careers of mathematicians?

The natural logarithm appears in physics, biology, sociology, economics and more. Students of physics know that many of the calculations, for example in electrodynamics and quantum mechanics, would be impossible if it were not for the natural logarithm. The universal applicability of the natural logarithm suggests that it is something that exists in the world in which we live and therefore it is a characteristic of the natural world.

7.4 Integration techniques

Investigation 15

Conceptual understanding:

An understanding that certain questions require application of knowledge from different areas of mathematics will aid solving some more complicated problems.

1 Use antidifferentiation to evaluate $\int e^x dx$.

Answer: Since $\frac{d}{dx}\left(e^x\right) = e^x$ it follows that $\int e^x dx = e^x + c$

2 Find the value of $A_n = \int_0^{na} e^x dx$

Answer: $A_n = \int_0^{na} e^x dx = \left[e^x\right]_0^{na} = e^{na} - 1$

3 What is does A_n represent?

Answer: The area bounded by the curve $y = e^x$, the x-axis and the lines $x = 0$ and $x = na$

.

4 Let $I_n = \int_{(n-1)a}^{na} e^x dx$. Copy and complete the following table.

Answer:

n	$\mathbf{I_n = \int_{(n-1)a}^{na} e^x dx}$
1	$I_1 = \int_0^a e^x dx = e^a - 1$
2	$I_2 = \int_a^{2a} e^x dx = e^{2a} - e^a = e^a\left(e^a - 1\right)$
3	$I_3 = \int_{2a}^{3a} e^x dx = e^{3a} - e^{2a} = e^{2a}\left(e^a - 1\right)$
4	$I_4 = \int_{3a}^{4a} e^x dx = e^{4a} - e^{3a} = e^{3a}\left(e^a - 1\right)$
5	$I_5 = \int_{4a}^{5a} e^x dx = e^{5a} - e^{4a} = e^{4a}\left(e^a - 1\right)$

5 Show that the integrals $I_n = \int_{(n-1)a}^{na} e^x dx$ form a geometric sequence and state the first term and the common ratio.

Answer: From the table above it can be seen that the integrals $I_n = \int_{(n-1)a}^{na} e^x dx$ form a geometric sequence with first term $I_1 = e^a - 1$ and common ratio $r = e^a$

6 Evaluate $\sum_{r=1}^{n} I_r$.

Answer: sum of the first n terms of the series is given by:

$$I_1 + I_2 + \ldots + I_n = \frac{I_1\left(1-\left(e^a\right)^n\right)}{1-e^a} = \frac{\left(e^a - 1\right)\left(1 - e^{na}\right)}{1-e^a} = e^{na} - 1 = A_n$$

7 Write a general comment to summarize your results from part A.

Answer: If the area bounded by the curve, the x-axis and the lines $x = 0$ and $x = na$ is divided into intervals of constant width, a, the areas I_n of each interval will form a geometric sequence, irrespective of the length of a.

8 Use antidifferentiation to find $\int \frac{1}{x} dx$. Comment on any limitations.

Answer: $\frac{d}{dx}(\ln x) = \frac{1}{x} \Rightarrow \int \frac{1}{x} dx = \ln x + c$. Since $\ln x$ is only defined for $x > 0$ it follows that $\int \frac{1}{x} dx = \ln x + c$, for $x > 0$ or $\int \frac{1}{x} dx = \ln|x| + c$.

9 Find the value of $\int_1^{na} \frac{1}{x}\,dx, n \in \mathbb{Z}^+, n > 1$.

Answer: $B_n = \int_1^{na} \frac{1}{x}\,dx = [\ln x]_1^{na} = \ln na - \ln 1 = \ln na$

10 What is does B_n represent?

Answer: The area bounded by the curve $y = \frac{1}{x}$, the x-axis and the lines $x = 1$ and $x = na$.

11 Copy and complete the following table:

Answer:

n	**H_n**
1	$H_1 = \int_1^{a} \frac{1}{x}\,dx = \ln a - \ln 1 = \ln a$
2	$H_2 = \int_a^{2a} \frac{1}{x}\,dx = \ln 2a - \ln a = \ln 2$
3	$H_3 = \int_{2a}^{3a} \frac{1}{x}\,dx = \ln 3a - \ln 2a = \ln\left(\frac{3}{2}\right)$
4	$H_4 = \int_{3a}^{4a} \frac{1}{x}\,dx = \ln 4a - \ln 3a = \ln\left(\frac{4}{3}\right)$
5	$H_5 = \int_{4a}^{5a} \frac{1}{x}\,dx = \ln 5a - \ln 4a = \ln\left(\frac{5}{4}\right)$

12 Find the value of $\sum_{r=1}^{n} H_r$.

Answer: $\sum_{r=1}^{n} H_r = \ln a + \ln 2 + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) + \ldots + \ln\left(\frac{n-1}{n-2}\right) + \left(\frac{n}{n-1}\right)$

$$= \ln\left(a \times 2 \times \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \ldots \times \frac{n-1}{n-2} \times \frac{n}{n-1}\right) = \ln(a \times n) = \ln na$$

13 Write a general comment to summarize your results from part B.

Answer: If the area bounded by the curve $y = \frac{1}{x}$, the x-axis and the line $x = na$ is divided into intervals of constant width, a, the subdivisions will be areas that follow a pattern which makes it easy to calculate the sum using the rules of logarithms.

14 Factual: What two areas of mathematics have you combined in answering this investigation?

Answer: Integration (Calculus) and geometric series (Number)

15 Conceptual: How does the ability to draw from your understanding of various aspects of mathematics help solve certain more complicated problems?

Answer: An understanding that certain questions require application of knowledge from different areas of mathematics will aid solving some more complicated problems.

Investigation 16

Conceptual understanding:

Antidifferentiation and the inverse chain rule help to find integrals of composite exponential functions and reciprocal functions.

1 a $y = e^{ax+b}$

Answer: $\frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$

b $y = \ln(ax+b)$

Answer: $\frac{d}{dx}(\ln(ax+b)) = \frac{a}{(ax+b)}$

2 a $\int e^{ax+b}dx$

Answer: $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$

b $\int \frac{1}{(ax+b)}dx$

Answer: $\int \frac{1}{(ax+b)}dx = \frac{1}{a}\ln|ax+b| + c$

3 Use antidifferentiation to determine $\int a^x dx$.

Answer: $\frac{d}{dx}\left(a^x\right) = (\ln a)a^x$

$$\Rightarrow \int a^x dx = \frac{1}{\ln a}\int (\ln a)a^x dx = \frac{a^x}{\ln a} + c$$

4 Use antidifferentiation to determine $\int \frac{f'(x)}{f(x)}dx$.

Answer: $\frac{d}{dx}\left(\ln\left(f(x)\right)\right) = \frac{1}{f(x)} \times f'(x)$

$$\Rightarrow \int \frac{f'(x)}{f(x)}dx = \ln\left(f(x)\right) + c$$

5 **Conceptual:** What techniques have you used in this investigation to help you find the integrals of composite exponential functions and reciprocal functions?

Answer: Antidifferentiation and the inverse chain rule help to find integrals of composite exponential functions and reciprocal functions.

TOK

Mathematics may not be the language of the universe, but rather the language/logical system the brain uses to analyze and respond to the universe. Ask students what they think about this and to consider where the real foundations of mathematics originate.

You might want to use facts such as The Higgs Boson was predicted with the same tool as the planet Neptune and the radio wave: with mathematics, which might mean that our universe isn't just described by mathematics, but that it is mathematics in the sense that we're all parts of a giant mathematical object.

Max Tegmark and Brian Butterworth provide some interesting insights and contrasting opinions on YouTube.

TOK

Consider the number e or logarithms, did they already exist before people defined them? This topic is an opportunity for teachers to generate reflection on "the nature of mathematics".

Investigation 17

Conceptual understanding:

Manipulating an integrand to be in a certain form allows you find an appropriate substitution and integrate the expression.

1 Use an appropriate u-substitution to work out the indefinite integral $\int \frac{1}{\sqrt{4-x^2}}\,dx$.

Answer: Let $x = 2\sin u \Rightarrow dx = 2\cos u\,du$

$$\Rightarrow I = \int \frac{2\cos u}{\sqrt{4-4\sin^2 u}}\,du = \int du = u + c \;\Rightarrow I = \arcsin\left(\frac{x}{2}\right) + c$$

2 Hence find the value of the definite integral $I_D = \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$.

Answer: $I_D = \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\arcsin\left(\frac{x}{2}\right)\right]_0^1 = \arcsin\left(\frac{1}{2}\right) - \arcsin 0 = \frac{\pi}{6}$

3 What is the value of the substituted variable u at the lower and upper bounds of x?

Answer: $x = 2\sin u \Rightarrow u = \arcsin\left(\frac{x}{2}\right)$

$$x = 1 \Rightarrow u = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$x = 0 \Rightarrow u = \arcsin(0) = 0$$

4 Re-write I_D, including the upper and lower bounds, purely in terms of the variable u. Find the value of this definite integral.

Answer: $I_D = \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{2\cos u}{\sqrt{4-4\sin^2 u}}\,du = \int_0^{\frac{\pi}{6}} du \;= [u]_0^{\frac{\pi}{6}} = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

5 What do you notice about your answers to question **2** and question **4**?

Answer: You obtain the same value for the definite integral, whether you

Work out the indefinite integral in terms of the substituted variable *u*, then substitute back to the original variable *x* and evaluate the limits, *or*

Change the limits so that they are given in terms of *u*, then perform the whole definite integration in terms of *u*.

i.e. $\int_{x=0}^{x=1} \frac{2\cos u}{\sqrt{4-4\sin^2 u}}\,du = \left[\arcsin\frac{x}{2}\right]_0^1 \equiv \int_0^{\frac{\pi}{6}} \frac{2\cos u}{\sqrt{4-4\sin^2 u}}\,du = [u]_0^{\frac{\pi}{6}}$

6 Repeat questions **1** to **5** for $I = \int \frac{1}{\sqrt{4-9x^2}}\,dx$ and $I_D = \int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\,dx$.

Answer: $I = \int \frac{1}{\sqrt{4-9x^2}}\,dx$

Let $3x = 2\sin u \Rightarrow 3 = 2\cos u\frac{du}{dx} \Rightarrow dx = \frac{2}{3}\cos u\,du$

$$I = \int \frac{1}{\sqrt{4-9x^2}}\,dx = \int \frac{\frac{2}{3}\cos u}{\sqrt{4-4\sin^2 u}}\,du = \int \frac{1}{3}\,du \;= \frac{1}{3}u + c = \frac{1}{3}\arcsin\left(\frac{3x}{2}\right) + c$$

Method 1: Switching back to the original variable *x* to find I_D:

$$I_D = \int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\,dx = \left[\frac{1}{3}\arcsin\left(\frac{3x}{2}\right)\right]_0^{\frac{1}{3}}$$

$$= \left(\frac{1}{3}\arcsin\left(\frac{1}{2}\right)\right) - \left(\frac{1}{3}\arcsin(0)\right) = \frac{\pi}{18}$$

Method 2: Finding the upper and lower bounds in terms of the new variable *u* to find I_D:

$$u = \arcsin\left(\frac{3x}{2}\right)$$

$x = \frac{1}{3} \Rightarrow u = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ $\qquad$ $x = 0 \Rightarrow u = \arcsin(0) = 0$

$$I_D = \int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{3}\,du = \left[\frac{1}{3}u\right]_0^{\frac{\pi}{6}} = \frac{\pi}{18} - 0 = \frac{\pi}{18}$$

7 Now repeat questions **1** to **5** for the general integrals $I = \int \frac{1}{\sqrt{a^2 - b^2x^2}} \mathrm{d}x$ and

$$I_D = \int_0^{\frac{a}{2b}} \frac{1}{\sqrt{a^2 - b^2x^2}} \mathrm{d}x$$

Answer: $I = \int \frac{1}{\sqrt{a^2 - b^2x^2}} \mathrm{d}x$

$$\text{Let } bx = a\sin u \Rightarrow b = a\cos u \frac{\mathrm{d}u}{\mathrm{d}x} \Rightarrow \mathrm{d}x = \frac{a}{b}\cos u \mathrm{d}u$$

$$I = \int \frac{1}{\sqrt{a^2 - b^2x^2}} \mathrm{d}x = \int \frac{\frac{a}{b}\cos u}{\sqrt{a^2 - a^2\sin^2 u}} \mathrm{d}u = \int \frac{1}{b} \mathrm{d}u$$

$$= \frac{1}{b}u + c = \frac{1}{b}\arcsin\left(\frac{bx}{a}\right) + c$$

Method 1: Switching back to the original variable x to find I_D:

$$I_D = \int_0^{\frac{a}{2b}} \frac{1}{\sqrt{a^2 - b^2x^2}} \mathrm{d}x = \left[\frac{1}{b}\arcsin\left(\frac{bx}{a}\right)\right]_0^{\frac{a}{2b}}$$

$$= \left(\frac{1}{b}\arcsin\left(\frac{1}{2}\right)\right) - \left(\frac{1}{b}\arcsin(0)\right) = \frac{\pi}{6b}$$

Method 2: Finding the upper and lower bounds in terms of the new variable u to find I_D:

$$bx = a\sin u \Rightarrow u = \arcsin\left(\frac{bx}{a}\right)$$

$$x = \frac{a}{2b} \Rightarrow u = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$x = 0 \Rightarrow u = \arcsin(0) = 0$$

$$I_D = \int_0^{\frac{a}{2b}} \frac{1}{\sqrt{a^2 - b^2x^2}} \mathrm{d}x = \int_0^{\frac{\pi}{6}} \frac{1}{b} \mathrm{d}u = \left[\frac{1}{b}u\right]_0^{\frac{\pi}{6}} = \frac{\pi}{6b} - 0 = \frac{\pi}{6b}$$

8 Find an appropriate u-substitution for the indefinite integral $I = \int \frac{1}{a^2 + b^2x^2} \mathrm{d}x$ and find I in terms of x. (Hint: Remember the identity 1+ tan $2x$ = sec $2x$.)

Answer: $I = \int \frac{1}{a^2 + b^2x^2} \mathrm{d}x$

$$\text{Let } bx = a\tan u \Rightarrow b = a\sec^2 u \frac{\mathrm{d}u}{\mathrm{d}x} \Rightarrow dx = \frac{a}{b}\sec^2 u\mathrm{d}u$$

$$I = \int \frac{1}{a^2 + b^2x^2}\,\mathrm{d}x = \int \frac{\frac{a}{b}\sec^2 u}{a^2 + a^2\tan^2 u}\,\mathrm{d}u$$
$$= \int \frac{1}{ab}\,\mathrm{d}u = \frac{1}{ab} \times u + c = \frac{1}{ab}\arctan\left(\frac{bx}{a}\right) + c$$

9 How could you rearrange the denominator in the integrals $\int \frac{1}{x^2 - 2x + 5}\,\mathrm{d}x$ in order to find an appropriate trigonometric u-substitution?

Answer: By writing the denominator as a sum of two squares, you can determine the integral using a substitution similar to the one in question **8**.

$$x^2 - 2x + 5 = x^2 - 2x + 1 + 4 = (x-1)^2 + 2^2$$

10 Evaluate $\int_3^{1+2\sqrt{3}} \frac{1}{x^2 - 2x + 5}\,\mathrm{d}x$

Answer: $\int_3^{1+2\sqrt{3}} \frac{1}{x^2 - 2x + 5}\,\mathrm{d}x = \int_3^{1+2\sqrt{3}} \frac{1}{(x-1)^2 + 2^2}\,\mathrm{d}x$

Let $x - 1 = 2\tan u \Rightarrow dx = 2\sec^2 u\,du$

$$\int_3^{1+2\sqrt{3}} \frac{1}{x^2 - 2x + 5}\,\mathrm{d}x = \int_{x=3}^{x=1+2\sqrt{3}} \frac{2\sec^2 u}{4\tan^2 u + 4}\,\mathrm{d}u$$
$$= \int_{x=3}^{x=1+2\sqrt{3}} \frac{1}{2}\,\mathrm{d}u = \left[\frac{1}{2}u\right]_{x=3}^{x=1+2\sqrt{3}} = \left[\frac{1}{2}\arctan\left(\frac{x-1}{2}\right)\right]_3^{1+2\sqrt{3}}$$
$$= \left(\frac{1}{2}\arctan\left(\frac{1+2\sqrt{3}-1}{2}\right)\right) - \left(\frac{1}{2}\arctan\left(\frac{3-1}{2}\right)\right)$$
$$= \frac{1}{2}\left(\arctan\sqrt{3} - \arctan 1\right) = \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{24}$$

Alternatively:

$$\int_2^{1+2\sqrt{3}} \frac{1}{x^2 - 2x + 5}\,\mathrm{d}x = \int_2^{1+2\sqrt{3}} \frac{1}{(x-1)^2 + 2^2}\,\mathrm{d}x$$

Let $x - 1 = 2\tan u \Rightarrow \mathrm{d}x = 2\sec^2 u\,du$

$$\Rightarrow u = \arctan\left(\frac{x-1}{2}\right)$$

$$x = 1 + 2\sqrt{3} \Rightarrow u = \arctan\left(\frac{1+2\sqrt{3}-1}{2}\right) = \arctan\sqrt{3} = \frac{\pi}{3}$$

$$x = 3 \Rightarrow u = \arctan\left(\frac{3-1}{2}\right) = \arctan 1 = \frac{\pi}{4}$$

$$\int_{3}^{1+2\sqrt{3}} \frac{1}{x^2 - 2x + 5}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sec^2 u}{4\tan^2 u + 4}\,du$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2}\,du = \frac{1}{2}\left[u\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{24}$$

11 Conceptual: How does manipulating the form in which an integral is presented allow you to integrate more easily?

Answer: Manipulating an integrand to be in a certain form allows you to find an appropriate substitution and integrate the expression.

TOK

Have students express their thoughts on what is imagination.

"Imagination is the beginning of creation. You imagine what you desire, you will what you imagine and at last you create what you will." George Bernard Shaw

Albert Einstein thought so. He said: "I'm enough of an artist to draw freely on my imagination, which I think is more important than knowledge.

Knowledge is limited. Imagination encircles the world."

When you see your students share the knowledge they have learned from you, don't you feel proud?

Now, when you see their imagination use that knowledge and take a step further, that's amazing.

Have students write about which they think is the more important, and why.

Investigation 18

This investigation does not have a conceptual understanding, but is intended to give students an understanding of how the integration by parts formula is derived.

1 How would you represent area 1 using integrals?

Answer: Area 1 = $\int_{v_1}^{v_2} u(v)\,dv$

2 If the axes are interchanged the function can be interpreted as $v(u)$. How would you represent area 2 using integrals?

Answer: Area 2 = $\int_{u_1}^{u_2} v(u)\,du$

3 What is the sum of your results?

Answer: Area 1 + area 2 $= \int_{v_1}^{v_2} u(v)\,dv + \int_{u_1}^{u_2} v(u)\,du$

4 What is the area of rectangle OFBD?

Answer: Area of rectangle OFBD = u_2v_2

5 What is the area of rectangle OEAC?

Answer: Area of rectangle OEAC= u_1v_1

6 Express area 1 in terms of the areas of rectangles OFBD and OEAC and your expression for area 2 from question **2**.

Answer: Area 1 = $(u_2v_2 - u_1v_1) - \int_{u_1}^{u_2} v\,\mathrm{d}u$

TOK

Shared knowledge consists of belief and practices which are communicated to other people and might well need translation by the receiver. The receiver might need enough skill to understand what is being transmitted and there are many famous cases of mistaken knowledge such as the charge of the light brigade.

An individual's knowledge claims might be distorted by peer pressure, fantasy or bias.

Investigation 19

Conceptual understanding:

The cyclical nature of derivatives of $\sin x$ and $\cos x$, and the invariance of the exponential function under differentiation and integration allows us to integrate the product of an exponential function with a trigonometric function.

1 Choosing either function to differentiate and the other to integrate, use integration by parts once on I.

Answer:

Let $u = e^x \Rightarrow \frac{du}{dx} = e^x$

$\frac{dv}{dx} = \sin x \Rightarrow v = \int \sin x dx = -\cos x$

$$I = \int e^x \sin x \, dx$$
$$= -e^x \cos x - \int -e^x \cos x \, dx$$
$$= -e^x \cos x + \int e^x \cos x \, dx$$

Or, let

$u = \sin x \Rightarrow \frac{du}{dx} = \cos x$

$\frac{dv}{dx} = e^x \Rightarrow v = \int e^x dx = e^x$

$$I = \int e^x \sin x \, dx$$
$$= e^x \sin x - \int e^x \cos x \, dx$$

2 Use integration by parts again on the integral which remains in each of the results from question **1**. Each time, make sure you differentiate the same function as before (i.e. if you differentiated the exponential function the first time, differentiate that again; if you differentiated the trigonometric function the first time, differentiate the trigonometric function again).

Answer:

$\mathrm{I} = \int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$ $u = e^x \Rightarrow \frac{du}{dx} = e^x$ $\frac{dv}{dx} = \cos x \Rightarrow v = \int \cos dx = \sin x$ $\Rightarrow \mathrm{I} = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$	$\mathrm{I} = \int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$ $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$ $\frac{dv}{dx} = e^x \Rightarrow v = \int e^x dx = e^x$ $I = e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) dx\right)$ $= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

3 **Factual:** What do you notice about each of the expressions you obtained for I in question **2**?

Answer: Whichever variable you choose to differentiate / integrate, you are left with the same expression $I = e^x \sin x - e^x \cos x - \int e^x \sin x dx$

4 **Conceptual:** How can you obtain an expression for I which does not contain an integral?

Answer: By substituting I for $\int e^x \sin x dx$ in the integrated expression, you can use algebra to find an expression for I:

$$I = e^x \sin x - e^x \cos x - I$$

$$\Rightarrow 2I = e^x \sin x - e^x \cos x \qquad \Rightarrow I = \frac{e^x}{2}(\sin x - \cos x)$$

5 Use the method above to find $I = \int e^x \cos x dx$.

Answer: $I = \int e^x \cos x dx$

Let $u = e^x \Rightarrow \frac{du}{dx} = e^x$

$\frac{dv}{dx} = \cos x \Rightarrow v = \int \cos x dx = \sin x$

$$I = \int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

Again, let $u = e^x \Rightarrow \frac{dv}{dx} = e^x$

$\frac{dv}{dx} = \sin x \Rightarrow v = \int \sin x dx = -\cos x$

$$I = \int e^x \cos x dx = e^x \sin x - \left(-e^x \cos x - \int e^x (-\cos x) dx\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x (\cos x) dx$$

$$\Rightarrow I = e^x \sin x + e^x \cos x - I \qquad \Rightarrow I = \frac{e^x}{2}(\sin x + \cos x)$$

6 **Conceptual:** What properties of exponential and trigonometric functions allowed you to integrate the product of an exponential function with a trigonometric function?

Answer: The cyclical nature of derivatives of $\sin x$ and $\cos x$, and the invariance of the exponential function under differentiation and integration allows us to integrate the product of an exponential function with a trigonometric function.

TOK

You might want to read from the Gifford Lectures.

"Believe that" means that you think something is real. Belief can be a personal opinion and it means that you think that it is more concrete.

"Believe in" means that what you believe in is more personal and less concrete. Believe in is more hopeful and gives a sense of fulfilment.

Have students share several examples of each with a partner.

How does the teacher's shared knowledge fit into this? Is it something that students "believe that", "believe in" or trust?

How does trust play a role?

Developing inquiry skills

In the opening scenario for this chapter, you were asked to find the function *f* that models the height or the hydrangea bush for $t \geq 20$ and to show that the height of the hydrangea bush does not exceed 1.5 m. You should now be able to answer these questions.

Answer: $f(t) = -\frac{1}{5}e^{-\frac{t}{10}+3.09} + 1.1956878$; as t approaches infinity, $f(t)$ approaches 1.1956878 which is less than 1.5

Return to the opening problem. How have the skills you have learned in this chapter helped you to solve the problem?

Answer: To find the function for the height of the plant for $t > 20$ you needed to be able to integrate the composite function e^{ax+b}. This was a skill learned in the chapter.

A passing fad?

Approaches to Learning: Communication, Research

Exploration Criteria: Mathematical presentation, Reflection (D), Use of mathematics (E)

IB Topic: Exponentials and Logarithms

Introduction

This task gives further practice to students in finding data and using it to model (in this case exponential functions) and then reflecting on the usefulness of the model for predictions. Students could model by hand or use technology (or both). Students are not penalized for using technology if they can demonstrate understanding of the process that is being used.

$$45 = b^5$$
$$b = \sqrt[5]{45}$$
$$b = 2.14 \text{ (3sf)}$$
$$a = \frac{1}{2.14}$$
$$a = 0.467 \text{ (3 sf)}$$
$$\therefore P = 0.487(2.14)^t$$

At the time of writing, Fortnite is a massive phenomenon and the task here is to see whether this is a passing fad or if the exponential growth continues. Similar discussions could be had regarding 'the latest fad' and if data could be found then this could form the basis of the discussions for the task at the end.

Depending on where this is covered, during the chapter and previously you may wish to work through the calculations of the 'by hand' model and the model using technology. Students can then use this for their own data or, if they are able to, they could then devise these models themselves.

There are opportunities in the task for discussing the importance for consistent notation in an IA as well as Reflection (Criterion D) on the reliability of data that they can find.

Look at the data

The data is taken from the press releases of the developers, Epic Games, but will supposedly be subject to checks and scrutiny by rival companies.

The data is to the nearest million, so is not particularly accurate.

The data is of users but some of these users may play frequently and some may play only once in the time period being measured.

The dates are not very accurate - you assume the data is all released at the same point in the month.

Daily Average Users (DAU) or Monthly Average Users (MAU) or amount of time spent on the game may be more interesting information to collect if it is possible to find.

The growth looks exponential. This could be due to word of mouth with people suggesting to friends, etc, to play the game.

Model the data

The model could be useful in terms of predicting future advertising prices and revenues for the company or for rival companies to consider when a game may be reaching saturation point.

Emphasize the importance of making sure that the variables in the model are clearly defined.

Look at these models (or equivalent using the available software) with students:

By hand	**By technology** **(using TI-NSPIRE)**	**By technology** **(using Desmos)**
Choose 2 points (t, P) For example, (1,1) and (6,45). Substitute into the	Enter the data into a table. Label x list t and y list P. Menu > Statistics > A: Exponential Regression. Select x list as t and y list as P.	The data can be inputted into a table in Desmos and then a function can be found using regression by inputting the function $y_1 \quad a \cdot b^x$.

equation: $P = a \cdot b^t$ $1 = a \cdot b^1$ $\therefore a \cdot b = 1$ $a = \frac{1}{b}$ and $45 = a \cdot b^6$ So Draw the curve. Note: if you choose 2 different points you would get a different curve.	Press OK. Therefore, the equation of the exponential curve that best fits the data according to the GDC is $P = 1.81 \cdot 1.56^t$.	The equation of the exponential curve that best fits the data according to Desmos is $P = 1.81 \cdot 1.56^t$. Make sure that Desmos is in 'log mode' to obtain the same result. If log mode is not ticked then a different equation will be found. $P = 8.65 \cdot 1.27^t$ (this may be an opportunity to discuss/explore residuals. By clicking residuals 'plot' this gives a good visual measure of how accurate the model is).

The 'by hand' model only uses 2 points. The curve will naturally pass through these 2 points but is not necessarily close to any of the other points.

You could also choose a different function of a similar form, say $P = a \cdot b^t + c$, and find parameters *a*, *b* and *c* for the model. This is not a function that is available as an option on the GDC but you could use Desmos, for example, to find it.

If you were to find a model $P = a \cdot b^t + c$ by hand you would require 3 points to find the 3 variables.

You need to calculate how many months it is since July 2107. Substitute a value. The result is likely to be very large and discussion will be around the fact that the game will have reached saturation or a new game will have come along, etc.

Students could research the number of Fortnite players there are in the current month.

You could ask:

- *Is Fortnite still a popular game now?*

Students could compare this figure with their prediction based on their model. How big is the error? What does this tell you about the reliability of your previous model?

This will hopefully support the above.

Plot a new graph with the updated data you have found and try to fit another function to this data. Will a modified exponential model be a good fit? If not, what other function would be a better model that could be used to predict the number of users now?

This could be a good opportunity for a discussion around a logistic model that may be more appropriate. This from Khan Academy (https://goo.gl/KMmFbC) is a good summary of what happens when an exponential model is constrained by real life.

Extension

Hopefully there are numerous examples of 'the next big thing'. Good sources will be social media sites, games, technology uptake, etc.

Students should be encouraged to find their own data, display and model it.

This task could be written up as a mini-exploration perhaps assessed against a smaller number of criteria.

Here is a possible structure for this:

Mini-exploration

Write a brief exploration on what you find out.

This exploration should be between one and two pages depending on the number of diagrams/graphs that you use.

This is not an exercise in being able to copy from Wikipedia or other websites, but rather to find out relevant information and to rewrite it into an exploration.

Students could be marked against parts of the Criteria of the real Mathematics Exploration:

Criterion A: Presentation (3)

Your writing should be well-organised, coherent, logically developed and easy to follow. It includes an introduction, aim and conclusion.

Criterion B: Mathematical communication (3)

Use appropriate mathematical language and representation and define key terms.

Criterion D: Reflection (3)

You should review, analyse and evaluate your exploration. You should consider the significance of your findings, state possible limitations and/or extensions and make links to different fields and/or areas of mathematics.

Criterion E: Use of Mathematics (1)

Demonstrate that you fully understand the mathematics used in your exploration.

TOTAL (10)

8 Modelling change: more calculus

Essential understandings

Calculus describes rates of change between two variables. Understanding these rates of change allows us to model, interpret and analyze real-world problems and situations. Calculus helps us understand the behavior of functions and allows us to interpret the features of their graphs.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- The derivative may be represented physically as a rate of change and geometrically as the gradient or slope function.
- Areas under curves can be approximated by the sum of the areas of rectangles which may be calculated even more accurately using integration.
- Examining rates of change close to turning points helps to identify intervals where the function increases/decreases, and identify the concavity of the function.
- Numerical integration can be used to approximate areas in the physical world.
- Mathematical modelling can provide effective solutions to real-life problems in optimization by maximizing or minimizing a quantity, such as cost or profit.
- Derivatives and integrals describe real-world kinematics problems in two and three-dimensional space by examining displacement, velocity and acceleration.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
The area between two continuous functions in the interval $[a, b]$ that intersect at the end-points of the interval can be found by integrating the difference of the functions and taking the absolute value of the answer.	Investigation 1
Areas under curves can be approximated by the sum of the areas of rectangles which may be calculated even more accurately using integration.	Investigation 2
Geometrically, the volume of a solid of revolution evaluates the summation of infinitesimally thin cylinders that make up the cross-sections of the region.	Investigation 3

Integrating the velocity function between the endpoints of a given interval gives the displacement, and integrating the absolute value of the velocity function within the same bounds gives the total distance travelled.	Investigation 4
Mathematical modelling can provide effective solutions to real-life problems in optimization by maximizing or minimizing a quantity, such as cost or profit	Investigation 5
Derivatives and integrals describe real-world kinematics problems in two- and three-dimensional space by examining displacement, velocity and acceleration.	Investigation 6
An integrating factor is a function by which an ODE, in standard form, can be multiplied in order to be able to integrate the differential equation, and thereby solve it.	Investigation 7
The numerical approximation to the graph of the differential equation can be made stronger by selecting a smaller value of the step size, *h*.	Investigation 8
When the limit of a function in quotient form, as *x* approaches *a*, produces indeterminate forms, then take the limit as *x* approaches *a* of the derivative of each function i.e. $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$.	Investigation 9
A finite number of terms of an infinite series can be a general approximation of a function over a limited domain.	Investigation 10

Syllabus sections covered in this chapter:

- SL5.9
- SL5.11
- AHL5.13
- AHL5.17
- AHL5.18
- AHL5.19

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 519: Modelling change: more calculus	Page 526: Example 6 Page 534: Example 10 Page 544: Example 16 Page 548: Example 18 Page 559: Example 26	Page 521: Example 2 Page 529: Example 7	Pages 527, 532, 549, 562

Assessment opportunities

End of chapter test	Chapter review	Exam practice
Page 563	Page 565	Page 566

8.1 Areas and volumes

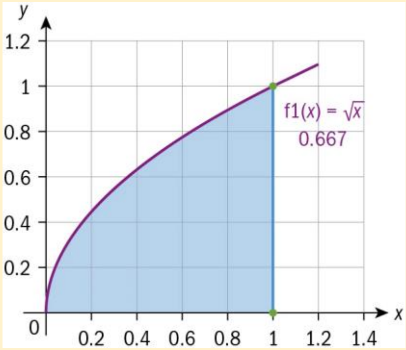
Investigation 1

Conceptual understanding:

The area between two continuous functions in the interval [*a, b*] that intersect at the end-points of the interval can be found by integrating the difference of the functions and taking the absolute value of the answer.

1 Sketch the graph of $y = \sqrt{x}$, and shade the area between the curve, the x-axis, and the line $x = 1$. Find the shaded area.

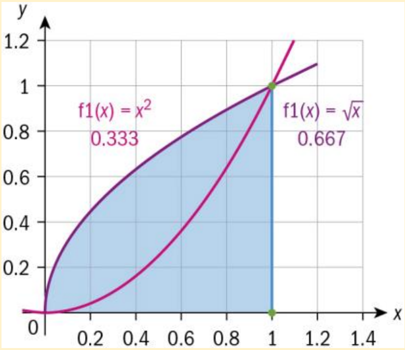
Answer:



Shaded area = 0.667 or $\frac{2}{3}$

2 On the same axes, draw the graph of $y = x^2$ and shade the area between the curve, the x-axis, and the lines $x = 0$ and $x = 1$

Answer:



Shaded area under $y=x^2$ is 0.333 or $\frac{1}{3}$

3 Find the area between the curves $y = \sqrt{x}$ and $y = x^2$, and explain how you found this area.

Answer: Subtracting the two areas leaves the area of the region between the two curves, $\frac{1}{3}$or 0.333 square units.

4 Find the value of the definite integral $\int_a^b [f(x) = g(x)]\,\mathrm{d}x\int$, where $x = a$ and $x = b$ are the points of intersection of the curves.

Compare your answer to the value you found in question **3** for the area between the curves. What do you notice?

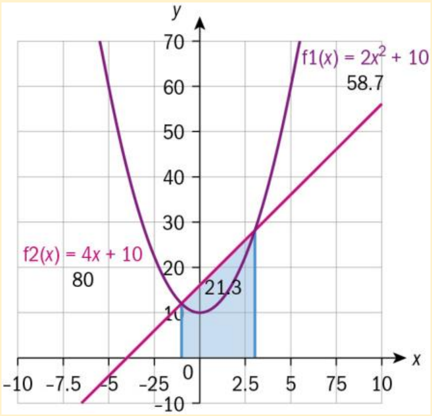
Answer: $f(x) = \sqrt{x}; g(x) = x^2;\ \int_0^1 x^{\frac{1}{2}} - x^2\,\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$; value of this definite integral is the same as the value obtained for the area between the curves.

5 **a** Find the area bounded by the curve $y = 2x^2 + 10$, the x-axis, and the lines $x = -1$ and $x = 3$.

b Find the area bounded by the line $y = 4x+16$, the x-axis, and the lines $x = -1$ and $x = 3$.

c Using your answers to parts **a** and **b**, find the area between the curve $y = 2x^2+10$ and the line $y = 4x+16$

Answer:



a 80

b 58.7

c Area between curves = 80 – 58.7 = 21.3

6 Find the value of the definite integral $\int_a^b [f(x) - g(x)]\,dx$, where $x = a$ and $x = b$ are the points of intersection of the curves.

Compare your answer to the value you found in question **5c** for the area between the line and the curve. What do you notice?

Answer: $f(x) = 4x + 16; g(x) = 2x^2 + 10$

$$\int_{-1}^{3} [4x + 16 - (2x^2 + 10)]dx = \frac{64}{3} \approx 21.3$$

7 If you mistakenly interchange the two functions in the integral, how would your answer change?

Answer: Your answer would be the negative of the value you obtained with the functions the other way around.

8 How could you change the integral so that the placement of the functions is irrelevant?

Answer: Place absolute value signs around the integral.

9 **Conceptual:** How can you use a single integral to find the area between two curves whose functions are *f* and *g*, that intersect at $x = a$ and $x = b$, and are continuous in the interval $[a, b]$?

Answer: The area of two continuous functions in the interval $[a, b]$ that intersect at the end-points of the interval can be found by integrating the difference of the functions and taking the absolute value of the answer.

TOK

What does this tell us about mathematical knowledge?

Gabriel's horn is an interesting paradox. You have a hollow object which has infinite length. It can be filled with paint, but if you try to cover the surface with that paint, there won't be enough. How does that make sense? What does your intuition tell you?
Some say that "When Gabriel blows the horn we all go to paradise". The archangel Gabriel is shared by Judaism, Islam and Christianity.

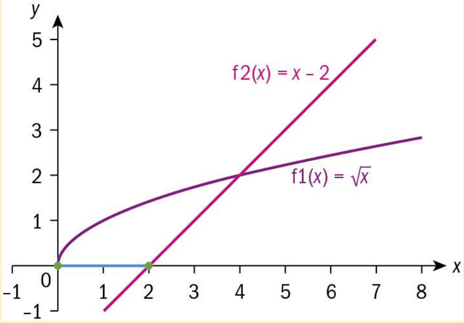
Investigation 2

Conceptual understanding:

Areas under curves can be approximated by the sum of the areas of rectangles which may be calculated even more accurately using integration.

1 Draw the graphs of the functions $y = \sqrt{x}$ and $y = x - 2$. Label *R* the region in the first quadrant which is bounded by the functions and the *x*-axis.

Answer:



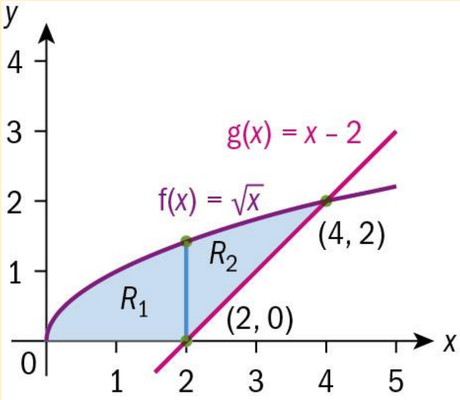
2 Why is it not possible to use the techniques you learned in Investigation **1** to find the area of the region R?

Answer: R is not entirely enclosed by the functions.

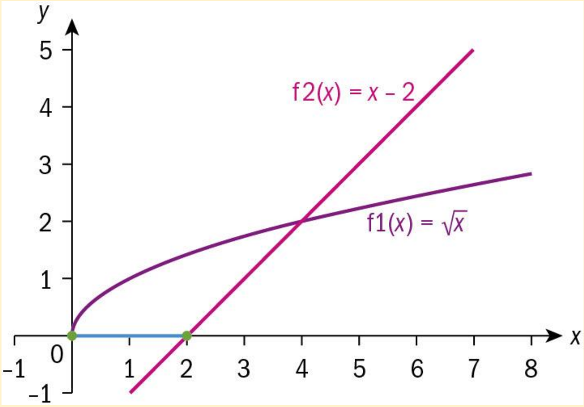
3 Divide R into two separate regions whose areas are possible to find with the integration techniques you have learned so far, and label them as R_1 and R_2.

Answer: There are two possible partitions, one involving a sum, and the other involving the difference:

Partition 1



Partition 2



4 Find the areas of R_1 and $R2$, and the area of the entire region.

Answer: $y = x - 2$ and $y = \sqrt{x}$ intersect when $\sqrt{x} = x - 2 \Rightarrow x = x^2 - 4x + 4$

$\Rightarrow 0 = x^2 - 5x + 4 \qquad \Rightarrow x = 1{,}4$

We discount the solution $x = 1$ as this only occurs for $y = 2 - x$

Using partition 1

$$\text{Area } R_1 = \int_0^2 \sqrt{x}\,\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2 = \frac{4}{3}\sqrt{2}$$

$$\text{Area } R_2 = \int_2^4 \sqrt{x} - (x-2)\,\mathrm{d}x$$

$$= \int_2^4 x^{\frac{1}{2}} + 2 - x\,\mathrm{d}x$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} + 2x - \frac{1}{2}x^2\right]_2^4$$

$$= \frac{16}{3} - \left(\frac{4}{3}\sqrt{2} + 2\right) = \frac{10 - 4\sqrt{2}}{3}$$

Area of $R = R_1 + R_2 = \dfrac{10}{3}$ or 3.33 sq. units

Using partition 2

$$R_1 = \int_0^4 \sqrt{x}\,\mathrm{d}x = \frac{16}{3} \quad R_2 = \int_2^4 x - 2\,\mathrm{d}x = 2 \quad R = R_1 - R_2 = \frac{10}{3}$$

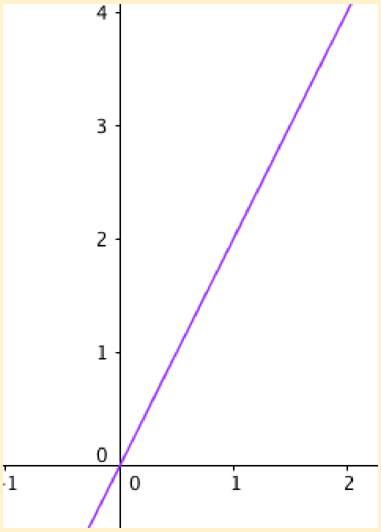
Investigation 3

Conceptual understanding:

Geometrically, the volume of a solid of revolution evaluates the summation of infinitesimally thin cylinders that make up the cross-sections of the region.

1 Sketch the graph of $y = 2x$ between $x = 0$ and $x = 2$.

Answer:



3 State the volume of one of the cylinders, in terms of y.

Answer: $V = \pi y^2 \mathrm{d}x$

4 Write the volume of the entire cone as the summation of the cylindrical cross-sections, in terms of *y*.

Answer: $V = \sum \pi y^2 \, \mathrm{d}x$. Note that we are summing the volumes of the many cylinders which lie in the range $0 < x < 2$.

5 Find the volume of the cone obtained by rotating the line *y* =2*x* in the interval [0, 2] through 2π radians about the *x*-axis.

Answer: $V = \pi\int_0^2 (2x)^2 \mathrm{d}x = 4\pi\left[\frac{x^3}{3}\right]_0^2 = \frac{32\pi}{3}$ cubic units

6 Compare your answer for the volume of the cone with the answer you would get using the formula for the volume of a cone, $V = \frac{\pi r^2 h}{3}$, where the radius of the base of the cone is 4, and the height is 2.

Answer: Using the formula, the volume is the same as the value you found in question **4**.

7 What is the radius, *x*, of each cylinder, in terms of *y*?

Answer: $x = \frac{y}{2}$

8 What is the volume of one of the cylindrical cross-sections?

Answer: $V = \pi y^2 \mathrm{d}y$

9 Write the volume of the entire cone as the summation of the cylindrical cross-sections, in terms of *x*.

Answer: $V = \sum \pi x^2 \, \mathrm{d}y$. Note that, this time, we are summing the volumes of the many cylinders which lie in the range $0 < y < 4$.

10 Find the volume of the cone obtained by rotating the line $x = \frac{y}{2}$ in the interval $0 \le y \le 4$ through 2π radians about the *y*-axis.

Answer: $V = \int_0^4 \pi\left[\frac{y}{2}\right]^2 \mathrm{d}y = \frac{16\pi}{3}$ cubic units

11 Compare your answer for the volume of the cone with the answer you would get using the formula for the volume of a cone.

Answer: Volume is the same as in question **9**, when the formula is used.

12 **Factual:** What is a solid of revolution?

Answer: The solid you obtain when you rotate a line or curve about the *x*-axis or *y*-axis.

13 **Conceptual:** Describe how you can calculate the volume of a solid of revolution.

Answer: Geometrically, the volume of a solid of revolution is the summation of infinitesimally thin cylinders that make up the cross-sections of the region.

8.2 Kinematics

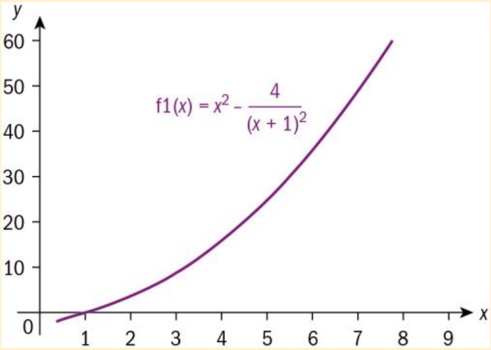
Investigation 4

Conceptual understanding:

Integrating the velocity function between the endpoints of a given interval gives the displacement, and integrating the absolute value of the velocity function within the same bounds gives the total distance travelled.

1 Sketch the graph of the particle's velocity for $0 \le t \le 6$.

Answer:



:

2 Find and interpret the initial velocity of the particle.

Answer: $v(0) = -4$ cm s^{-1}. The particle has an initial speed of 4 cm s^{-1} and is moving to the left.

3 Describe the motion of the particle during $t \in [0, 6]$.

Answer: After moving to the left, it then slows down and stops at $t = 1$ s. It then moves to the right with increasing speed, and reaches a maximum velocity of 35.9 cm s^{-1} at $t = 6$ s.

4 Find the displacement of the particle from $t = 0$ to $t = 1$.

Answer: $\int_0^1 v(t)\,dt = \int_0^1 \left[t^2 - \frac{4}{(t+1)^2}\right] dt$

$$= \left[\frac{t^3}{3} + \frac{4}{t+1}\right]_0^1 = -\frac{5}{3}\text{ cm}$$

5 The initial position of the particle is $s(0) = 5$ cm. Use the displacement from $t = 0$ to $t = 1$, and the initial position of the particle to find the position of the particle after the 1st second.

Answer: $s(0) = 5 \Rightarrow$ after 1st second, position is $5 - \frac{5}{3} = \frac{10}{3}$ cm.

6 Find the particle's final position.

Answer: $\int_0^6 v(t) = \int_0^6 \left[t^2 - \frac{4}{(t+1)^2}\right] dt = \left[\frac{t^3}{3} + \frac{4}{t+1}\right]_0^6 = 68.6$ cm

Final position is 68.6 +5=73.6 cm.

7 Using the initial position and the expression for $v(t)$, find an expression for $s(t)$, the particle's displacement at any time t.

Answer: $\int\left(t^2 - \frac{4}{(t+1)^2}\right)\mathrm{d}t = \frac{t^3}{3} + \frac{4}{t+1} + c$.

$s(0) = 5 \Rightarrow c = 1$

$s(t) = \frac{t^3}{3} + \frac{4}{t+1} + 1$

8 Use the displacement function you found in question **7** to find the position of the particle at t = 1 and t = 6. Compare your answers to those you found in questions **5** and **6**.

Answer: Using the expression for $s(t)$ from question **7**, $s(1)=\frac{10}{3}$ cm; $s(6)=73.6$ cm

These are the same as you found in questions **5** and **6**.

9 **Factual:** Describe the changes in displacement which occur over the 6 seconds of motion.

Answer: The particle's initial position was $s(0)=5$ cm. It then moved to the left by $-\frac{5}{3}$ cm, so was at $s(1)=\frac{10}{3}$ cm. It then moved to the right again, ending up at $s(6)$ = 73.6 cm.

10 Explain how you find the total distance the particle actually travelled in

the six seconds, and find this distance.

Answer: The particle travelled 71.9 cm.

To find this, integrate the positive values of every velocity of the particle in the given interval (that is, integrate the absolute value of the velocity function between t = 0 and t =6, i.e., $\int_0^6 \left|t^2 - \frac{4}{(t+1)^2}\right|\mathrm{d}t$). Calculate this integral and find the total distance travelled by the particle in the first six seconds.

11 **Factual:** What is the difference between displacement and distance?

Answer: Displacement is the overall change in position and distance is how much ground an object actually covered.

12 **Conceptual:** How can you integrate the velocity function to find the displacement of a particle over a given time interval? How can you adapt this method to instead find the total distance travelled?

Answer: Integrating the velocity function between the endpoints of a given interval gives the displacement, and integrating the absolute value of the velocity function within the same bounds gives the total distance travelled.

TOK

Students often ask "Why are we doing this?" with the pure mathematics sections and kinematics is a good real life application, but should kinematics be in science or mathematics?

You might want to ask: Who decides what is mathematics?

Using a cultural context to integrate science and mathematics teaching and learning using students' daily life experiences, cultural heritage can promote awareness, sensitivity, appreciation, respect, and pride in community and culture.

8.3 Ordinary differential equations (ODEs)

Investigation 5

Conceptual understanding:

Mathematical modelling can provide effective solutions to real-life problems in optimization by maximising or minimising a quantity, such as cost or profit

1 Write an equation relating an object's acceleration due to gravity on the moon with the rate of change of the object's velocity, *v*. Hence, find a formula for *v* using the initial condition that at $t = 0$, $v = 0$.

Answer: $a(t) = \frac{dv}{dt} \Rightarrow \frac{dv}{dt} = -1.625$

Hence $v(t) = \int -1.625 dt = -1.625t + c$;

Since $v(0)=0$, $v(t) = -1.625t$.

2 Write an equation relating the object's velocity, which you found in question **1**, with its rate of change of height above the ground, *s*. Use the initial height of the feather and hammer to find a formula for their height above the ground at any time *t*.

Answer: $\frac{ds}{dt} = -1.625t \Rightarrow s(t) = \int(-1.625t)dt = -\frac{1.625}{2}t^2 + c$.

Since $s = 1.22$ when $t = 0$, $c = 1.22$ and $s(t) = -0.8125t^2 + 1.22$.

3 How long did it take the two objects to reach the ground after the astronaut let them drop?

Answer: Objects reach the ground when $s = 0$.

$s(t) = -0.8125t^2 + 1.22 = 0 \Rightarrow t = 1.23\text{s}$

Reflect: Do you think that populations grow as in the model shown? That is, does a population grow exponentially for an indefinite period of time?

Answer: Due to factors such as biological, environmental, economical, etc., populations do not exhibit extended exponential growth, but rather at some point there will be a limiting value to the population.

Reflect: Compare the graph of the exponential model used in Example 11 to the graph of the logistic model shown here, which is used in Example 13. Why is the logistic model superior to the exponential model to model population growth?

Answer: The logistic model reflects population growth more realistically because it takes into account the factors which contribute to slowing down population growth over time.

TOK

Maybe there is a better measure than radians in terms of pi.

The number tau is not as well-known as pi, but became more popular when Michael Hartl wrote "The Tau Manifesto" and it continues to gain support.

Research tau and decide if you think it is superior to pi.
This shows you that as certain conventions come about in mathematics it is very hard to go back if the choice was not ideal. Mathematics would be much easier if we used τ, but we don't and to change would take a massive effort. Some also argue that there is no need for change.

It shows that we can have opinions in mathematics. There is a debate about what is best. This is normally seen in subjects like Human Sciences, Art, Ethics, and is perhaps not expected in Mathematics.

What does this argument tell us about the nature of Mathematics?

How is this different or similar to other Areas of Knowledge?

Investigation 6

Conceptual understanding:

Derivatives and integrals describe real-world kinematics problems in two- and three-dimensional space by examining displacement, velocity and acceleration.

1 Is it possible to solve this differential equation using previous methods

you have learned? Explain why, or why not.

Answer: No, as it is not possible to either integrate directly, or to separate the variables.

2 Show that this equation can be written as $\frac{dy}{dx} = 1 + \frac{y}{x}$.

Answer: Dividing through by x gives the result.

3 Rewrite the equation for $\frac{dy}{dx}$ in terms of a new variable v, where $v = \frac{y}{x}$ and v is a function in x. Solve for y and find $\frac{dy}{dx}$.

Answer: $\frac{dy}{dx} = 1 + v$

4 Since $v = \frac{y}{x}$, find another expression for $\frac{dy}{dx}$ using implicit differentiation.

Answer: $\frac{dy}{dx} = v + x\frac{dv}{dx}$

5 Equate the two expressions you found for $\frac{dy}{dx}$ in questions **3** and **4**. Simplify your answer.

Answer: $v + x\frac{dv}{dx} = 1 + v \Rightarrow \frac{dv}{dx} = 1 \text{or} \frac{dv}{dx} = \frac{1}{x}$

6 Solve the differential equation in question **5**.

Answer: $v = \ln x + c$

7 Substituting $v = \frac{y}{x}$ into the expression you obtained in question **6**, show that $y = x(\ln x + c)$.

Answer: $\frac{y}{x} = \ln x + c \Rightarrow y = x(\ln x + c)$

8 Using the initial condition that $y = -1$ when $x = 1$, find c and hence solve the differential equation for y.

Answer: $-1 = (\ln 1 + c) \Rightarrow c = -1$ and $y = x(\ln x - 1)$

Investigation 7

Conceptual understanding:

An integrating factor is a function by which an ODE, in standard form, can be multiplied in order to be able to integrate the differential equation, and thereby solve it.

1 Rewrite this equation in the form $x^2\frac{dy}{dx} + 2xy = 1$ for functions f and g.

Answer: $\frac{d\lfloor x^2y \rfloor}{dx} = 1$

2 Integrate, with respect to x, both sides of the equation you found in question **1**. Hence find an expression for y in terms of x.

Answer: $x^2y = x + c \Rightarrow y = \frac{1}{x} + \frac{c}{x^2}$

3 Multiply each term in (*) by $I(x)$.

Answer: $I(x)\frac{dy}{dx} + I(x)p(x)y = I(x)q(x)$

4 What can you say about the first term in your answer to question **3** and the first term in the product rule for $\frac{d}{dx}(Iy) = Iy' + I'y$?

Answer: These terms are the same.

5 Solve the differential equation $I(x)p(x) = \frac{dI}{dx}$ for $I(x)$ by separating the variables.

Answer: $\int \frac{1}{I}\,du = \int p(x)\,dx$, $\ln I = \int p(x)\,dx$, $I = e^{\int p(x)\,dx}$

6 Now re-write your answer to question **3** using the function $I(x)$ you found in question **5**.

Answer: $e^{\int p(x)\,dx}\frac{dy}{dx} + e^{\int p(x)\,dx}p(x)y = e^{\int p(x)\,dx}q(x)$

7 Use the reverse of the product rule to write the equation in question **6** with a single term on the left-hand side.

Answer: $\frac{d}{dx}\left(ye^{\int p(x)\,dx}\right) = e^{\int p(x)\,dx}q(x)$

8 **Factual:** What is the standard form for a differential equation?

Answer: $\frac{dy}{dx} + p(x)y = q(x)$

9 **Conceptual:** How does the integrating factor help you solve an ordinary differential equation?

Answer: An integrating factor is a function by which an ODE of the form $\frac{dy}{dx} + p(x)y = q(x)$ can be multiplied in order to be able to integrate the differential equation, and thereby solve it.

Investigation 8

Conceptual understanding:

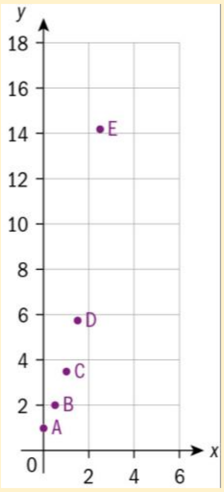
The numerical approximation to the graph of the differential equation can be made stronger by selecting a smaller value of the step size, *h*.

1

n	x_n	y_n	$f(x_n, y_n) = 1 + y_n$
0	0	1	2
1	0.5	1+2(0.5)=2	3
2	1	2+3(0.5)=3.5	4.5
3	1.5	3.5+4.5(0.5)=5.75	6.75
4	2	5.75+6.75(0.5)=9.125	10.125
5	2.5	9.125+10.125(0.5)=14.1875	15.1875

2 Plot the points (x_n, y_n) on the coordinate axes.

Answer:

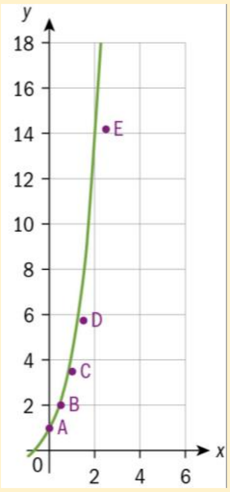


3 Select an appropriate analytical method and solve the differential equation $y' = 1 + y$.

Answer: Integrating factor method gives $y = 2e^{x} - 1$

4 n the same set of axes you used in question **2**, plot the graph of the function y you obtained by solving $y' = 1 + y$ in question **3**. Comment on your result.

Answer:



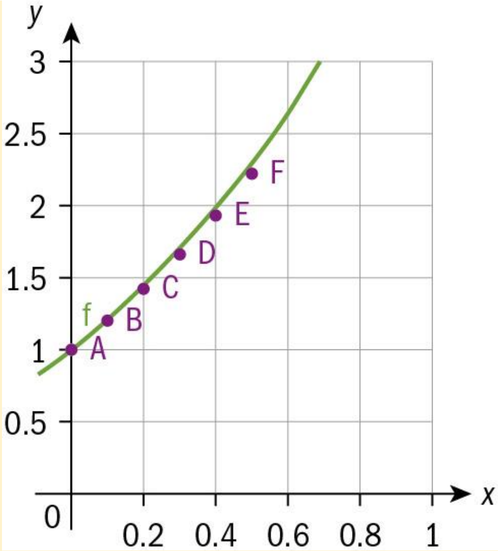
The error between the exact solution to $y = 2e^{x} - 1$, and the approximation given by the numerical method, increases as x increases.

5 Compare these values with the approximate values you obtained in question **1**.

Answer: The table of values confirms what the graphs in **4** show.

6

n	x_n	y_n	$\frac{dy}{dx} = f(x_n, y_n) = 1 + y$
0	0	1	2
1	0.1	1+(1+1)(0.1)=1.2	2.2
2	0.2	1.2+(1+1.2)(0.1)=1.42	2.42
3	0.3	1.662	2.662
4	0.4	1.928	2.928
5	0.5	2.221	3.221
6	0.6	2.543	3.543
7	0.7	2.897	3.897
8	0.8	3.287	4.287
9	0.9	3.716	4.716
10	1	4.187	5.187



7 How do your results show that varying the value of h in the numerical approximation of the solution to the differential equation helps to better approximate the solution of the differential equation at $x = 1$?

Answer: Using $h = 0.1$ gives a better approximation to the actual value at $x = 1$ than using $h = 0.5$.

8 **Conceptual:** How can a numerical approximation to the solution of an ODE be made stronger?

Answer: A numerical approximation of an ODE can be made stronger by taking smaller values of h.

8.4 Limits revisited

Investigation 9

Conceptual understanding:

When the limit of a function in quotient form, as x approaches a, produces indeterminate forms, then take the limit as x approaches a of the derivative of each function i.e.

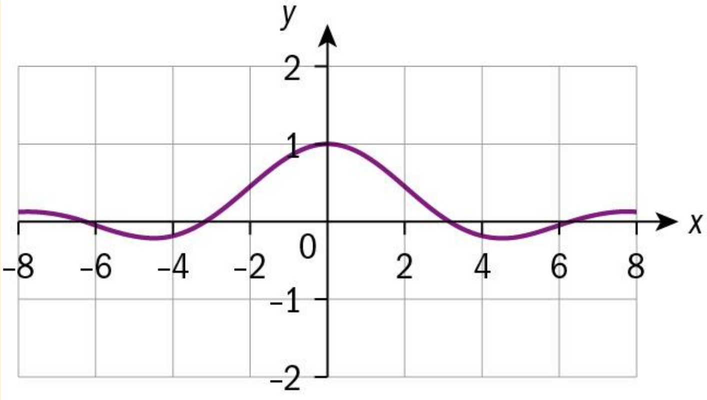
$\lim_{x \to 0} \frac{f'(x)}{g'(x)}$.

1 What do you obtain when you try to solve $\lim_{x \to 0} \frac{\sin x}{x}$ algebraically?

Answer: $\frac{0}{0}$

2 Graph the function $y = \frac{f(x)}{g(x)}$, and find graphically $\lim_{x \to 0} \frac{\sin x}{x}$. Justify your answer.

Answer:



$\lim_{x \to 0} \frac{\sin x}{x} = 1$ because the right and left hand limits at $x = 0$ are the same.

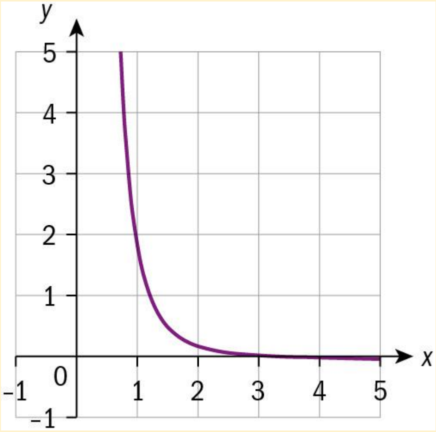
3 Find $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$, and compare your answer with $\lim_{x \to 0} \frac{\sin x}{x}$ from question **2**.

Answer: $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1$;

so in this case $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$

4 Repeat question **1** to show that you cannot algebraically find $\lim_{x \to \pi} \frac{1+\cos x}{x^2 \sin x}$, and repeat question **2** to find this limit graphically.

Answer: $\lim_{x \to \pi} \frac{1+\cos x}{x^2 \sin x} = \frac{1+\cos 0}{0^2 \sin 0} = \frac{1-1}{0} = \frac{0}{0}$



$\lim_{x \to \pi} \frac{1+\cos x}{x^2 \sin x} = 0$ because the right and left hand limits at $x=0$ are the same.

5 Find $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ algebraically, and compare it with $\lim_{x \to 0} \frac{f(x)}{g(x)}$ which you found graphically.

Answer: $\lim\limits_{x\to\pi}\dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to\pi}\dfrac{-\sin x}{2x\sin x + x^2\cos x}$

$$= \frac{-\sin\pi}{2\pi\sin\pi + \pi^2\cos\pi}$$

$$= \frac{0}{0-1} = 0$$

Again, for these functions, $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0}\dfrac{f'(x)}{g'(x)}$

6 Since $\lim\limits_{x\to 0}\dfrac{f'(x)}{g'(x)}$ is equivalent to $\dfrac{f'(0)}{g'(0)}$, write $\dfrac{f'(0)}{g'(0)}$ using the formal definition of the derivative of a function at $x = a$, $f'(a) = \lim\limits_{x\to 0}\dfrac{f(x)-f(a)}{x-a}$. Simplify your expression to show that $\dfrac{f'(0)}{g'(0)} = \lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$.

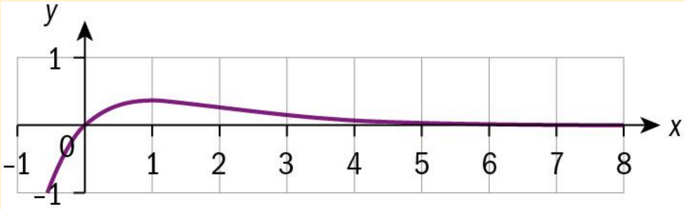
Answer: $\dfrac{f'(0)}{g'())} = \lim\limits_{x\to 0}\dfrac{\dfrac{f(x)-0}{x-0}}{\dfrac{g(x)-0}{x-0}} = \lim\limits_{x\to 0}\dfrac{f(x)-0}{g(x)-0} = \lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$.

7 What do you obtain when you try to find $\lim\limits_{x\to\infty}\dfrac{x}{e^x}$?

Answer: $\dfrac{\infty}{\infty}$

8 Graph the function in question **7** and write down its limit as $x \to \infty$.

Answer: 0



9 If $h(x) = x$ and $k(x)=e^x$, find $\lim\limits_{x\to\infty}\dfrac{h'(x)}{k'(x)}$.

Answer: $\lim\limits_{x\to\infty}\dfrac{h'(x)}{k'(x)} = 0$

10 Factual: What is an indeterminate form?

Answer: indeterminate form involves two functions whose limits cannot be determined from the limits of the individual functions.

11 Factual: How can you find an analytical solution to $\lim_{x\to\infty}\frac{h(x)}{k(x)}$ when substituting for x $= a$ results in the indeterminate forms $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$?

Answer: If taking the limit of a function of the form $y = \frac{f(x)}{g(x)}$ as x approaches a results in either of the indeterminate forms $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, find the limits of f' and g' at $x = a$. The limit is then $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$.

TOK

L'Hôpital's rule is named after the 17th-century French mathematician Guillaume de L'Hôpital, but the theorem was rumoured to be first discovered in 1694 by the Swiss mathematician Johann Bernoulli.

Research for the truth in the discovery and a possible transaction and then decide if it is ethically fair to name this theorem L'Hôpital's rule.

TOK

Opportunities for discussing hypothesis formation and testing, and then the formal proof can be tackled by comparing certain cases, through an investigative approach.

Investigation 10

Conceptual understanding:

A finite number of terms of an infinite series can be a general approximation of a function over a limited domain.

1 If $f(0)= P(0)$, find a_0.

Answer: $f(0) = \ln(1+0) = 0; P(0) = a_0 \Rightarrow a_0 = 0$

2 Find and equate the derivatives $f^{(n)}(x)$ and $P^{(n)}(x)$ for n =1, 2 ,3, 4, 5. Each time, solve $f^{(n)}(0)=P^{(n)}(0)$ and find the value of a_n. Do not simplify your answers.

Answer: $f'(x) = \frac{1}{1+x}; \mathrm{P}(0) = \mathrm{a}_1 + 2a_2x + 3a_3x^2 + 4a_4x^2 + 5a_5x^4$

$f'(0) = 1, P'(0) = a_1 \Rightarrow a_1 = 1$

$f''(x) = -1, P''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3$

$f''(0) = -1, P''(0) = 2a_2; 2a_2 = -1 \Rightarrow a_2 = -\frac{1}{2}$

$f'''(x) = \frac{2}{(1+x)^3}; P'''(x) = 6a_3 + 24a_4x + 60a_5x^2$

$$f'''(0) = 2; P'''(0) = 6\,a_3; 6a_3 = 2 \Rightarrow a_3 = \frac{2}{6}$$

$$f^{(4)}(x) = -\frac{6}{(1+x)^4}; P^{(4)}(x) = 24a_4 + 120a_5 x$$

$$f^{(4)}(0) = -6; P^{(4)}(0) = 24a_4; 24_4 = -6 \Rightarrow a_4 = -\frac{6}{24}$$

$$f^{(5)}(x) = \frac{24}{(1+x)^5}; P^{(5)}(x) = 120a_5$$

$$f^{(5)}(0) = 24; P^{(5)}(0) = 120a_5; 120a_5 = 24 \Rightarrow a_5 = \frac{24}{120}$$

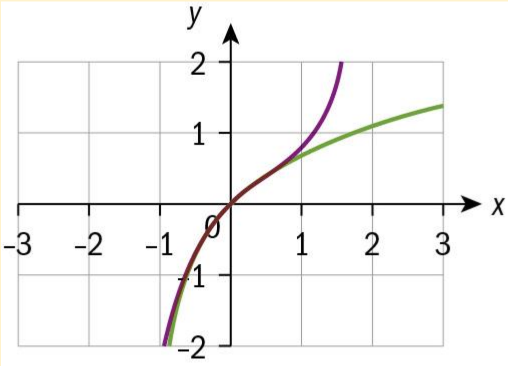
3 Find an expression for each coefficient an of the polynomial in terms of $f^{(n)}(0)$ and n.

Answer: $a_1 = \frac{f'(0)}{1!}; a_2 = \frac{f''(0)}{2!}; a_3 = \frac{f'''(0)}{3!}; a_4 = \frac{f^{(4)}(0)}{4!}; a_5 = \frac{f^{(5)}(0)}{5!}$

4 Now, simplify the coefficients and write out the polynomial.

Answer: $P(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$.

5 Graph $f(x)$ and $P(x)$ on the same set of axes, and describe the behaviour of the functions around an interval having $x = 0$ as its centre.

Answer: 

In the interval $]-1,1[$ the graphs of both functions are almost the same. This means that in this interval, the polynomial approximates the function.

6 **Conceptual:** How can a finite number of terms of an infinite series approximate a function?

Answer: A finite number of terms of an infinite series approximate a function?

TOK

An Einstein quote: Mathematical systems are invented, but it is a matter of discovery which of the various systems apply to reality. You can invent any formal system and prove theorems from axioms with complete certainty.

An axiom is a statement that is considered to be true and does not require a proof and is the starting point of reasoning. Axioms are then used to prove other statements using logical deduction. Euclidean Geometry is an example of an axiomatic system.

Be the particle!

Approaches to Learning/learner profile: Collaboration, Communication

Exploration Criteria: Personal engagement (C), Use of mathematics (E)

IB Topic: Calculus, Kinematics − motion in a straight line

In this task students get the opportunity to try and 'act out' the motion of a particle moving in a straight line given a velocity function from which they have found a displacement function and acceleration function. If the equipment (motion detector or filming equipment) is available they can then test how accurate they are. Otherwise it is possible to just generate discussions based around the questions posed.

The use of a motion detector or filming and then modelling on Logger pro would constitute good Personal engagement (Criterion C) in an exploration.

Motion of a particle in a straight **line**

Put the students into groups of 3 or 4.

If possible ensure that each group includes a mix of Physics students and non-Physics students as it is often the case that Physics students have encountered this topic before.

When $t = 0$, $v(0) = 10$ units/sec

The particle is stationary when the velocity is zero.

$(t - 2)(t - 5) = 0$, so the particle is stationary at $t = 2$ and $t = 5$

At $t = 6$ the particle is moving at 4 units/sec.

$s(t) < 0$ means that the particle is 5 units to the left of an origin.

$s(1)$ is the position of the particle after 1 second.

The displacement function $s(t)$ is found by integrating the velocity function and using the initial condition given to find the '+ c':

$$s(t) = \frac{t^3}{3} - \frac{7t^2}{2} + 10t - 5, 0 \le t \le 6$$

At $t = 6$ the particle is 1 unit to the right of the origin. ($s(6) = 1$)

The particle's displacement at each second of the journey is:

***t* (sec)**	0	1	2	3	4	5	6
$s(t)$ (units)	-5	$1\frac{5}{6}$	$3\frac{2}{3}$	$2\frac{1}{2}$	$\frac{1}{3}$	$-\frac{5}{6}$	1

Acceleration is the derivative of the velocity function. $a(t) = 2t - 7$

Initial acceleration is -7 units/s^2

The particle changes direction at $t = 2$ and $t = 5$.

Total distance travelled is the area under the velocity/time graph = 15 units.

Make sure that students produce the graphs correctly.

Remind students to include axes labels and a title.

Students can use their GDC to draw the graphs.

For **extension**, you could discuss, for example, the shape of each graph, the axes intercepts, the gradient at different parts on each graph, any maximum and minimum points, when each graph is increasing/decreasing, etc.

Be the particle for 6 seconds!

Give students the opportunity to discuss how they will walk the path of the particle.

You could ask:

- *What do you need?*
- *Who will do which job? etc…*

Before starting this activity, prepare by marking a scale from −5 to 5, where 0 is the origin, on the classroom board or wall or corridor etc.

If they have access to this equipment, to check the accuracy of the attempt, students could compare the motion with:

- a motion detector or
- a graphing programme like Logger pro.

You might want to discuss this equipment with the students:

A motion detector measures the time it takes for a high frequency sound pulse to travel from the detector to an object and back. This will determine the position of the object at different times.

The data can be fed into Logger Pro which can then use the change in position to calculate the object's velocity and acceleration which can be displayed either as a table or a graph.

If students do not have access to a motion detector, you could film some of the students' attempts from the side and analyse which one they think is the best using Logger Pro by inserting the movie they want to analyse. They can then set a scale and track the moving object (in this case pick a point on the 'particles' body that is easily visible in every shot). This will produce a graph of time (t) against displacement (s) from which the velocity and acceleration graphs can be found.

The comparison of the two cubic graphs will depend on the accuracy of the attempt.

To check how similar their cubic graph is to the cubic produced above, students can overlay a plot of the required displacement graph over the top to look for similarities.

To improve the model found, they can try again.

Extension

Make sure that students produce the graphs correctly.

Remind students to include axes labels and a title.

Students can use their GDC to draw the graphs.

Students could consider real-life applications of this motion in a straight line. For example:

They may like to discuss how they could determine the displacement, velocity and acceleration functions of an elevator or of a 100 m runner.

9 Modelling 3D space: vectors

Essential understandings

Geometry and trigonometry allows us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This topic provides us with the tool for analysis, measurement and transformation on quantities, movements and relationships.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- The properties of shapes depend on the dimension they occupy in space.
- Volume and surface area of shapes are determined by formulae, or general mathematical relationships or rules expressed using symbols or variables.
- The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.
- Equivalent measurement systems, such as degrees and radians, can be used for angles to facilitate ease of calculation.
- Different representations of the values of trigonometric relationships, such as exact or approximate, may not be equivalent to one another.
- The trigonometric functions of angles may be defined on the unit circle, which can visually and algebraically represent the periodic or symmetric nature of their values.
- Position and movement can be modelled in three-dimensional space using vectors.
- The relationships between algebraic, geometric and vector methods can help us to solve problems and quantify those positions and movements.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
A polygon with any sides parallel and equal in length results in its sides having equivalent directed line segments.	Investigation 2
The parallelogram and triangle laws are equivalent, and the circumstance may determine which one is used in a particular situation.	Investigation 3

The relationship between algebraic, geometric and vector methods can help us to solve problems and quantify those positions and movements.	

Syllabus sections covered in this chapter:

- AHL3.12
- AHL3.13
- AHL3.14
- AHL3.15
- AHL3.16
- AHL3.17
- AHL3.18

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 571: Modelling 3D space: vectors	Page 583: Example 7 Page 594: Example 16 Page 602: Example 23 Page 613: Example 33 Page 619: Example 39	Page 623: Example 41 Page 626: Example 43	Pages 585, 596, 604, 613, 620, 628, 633, 638

Assessment opportunities

End of chapter test	Chapter review	Exam practice
Page 638	Page 643	Page 645

9.1 Geometrical representation of vectors

Investigation 1

1 Why is it not possible for a triangle to have parallel sides?

Answer: The sum of all the angles in a triangle is 180°.

2 How many sides must a polygon have in order to have at least one pair of parallel sides?

Answer: Four sides (trapezium, parallelogram, rectangle, rhombus, square).

3 What are the simplest polygons that have parallel sides that have equal lengths?

Answer: Parallelogram, rectangle, rhombus and square.

4 What regular polygons have parallel sides?

Answer: Regular polygons with an even number of sides.

Investigation 2

Conceptual understanding:

A polygon with any sides parallel and equal in length results in its sides having equivalent directed line segments.

1 Answers will vary.

2 Join the vertices (draw diagonals) of both polygons. What do you notice?

Answer: Diagonals coincide with radii for a square and do not coincide for a regular pentagon.

3 Draw regular polygons with 6 to 9 sides and draw all the radii and only the longest diagonals for each polygon. What do you notice?

Answer: Diagonals coincide with radii for regular polygons with an even number of sides, and they do not coincide for regular polygons with an odd number of sides.

5 What do you notice about pairs of equivalent directed line segments in your diagrams?

Answer: In regular polygons with an even number of sides there are multiple groups of four equivalent directed line segments (two sides and two radii), while there are no equivalent directed line segments for regular polygons with an odd number of sides.

6 **Conceptual:** Can you make a conjecture connecting your observations to equivalent directed line segments?

Answer: A polygon with any sides parallel and equal in length results in its sides having equivalent directed line segments.

Investigation 3

Conceptual understanding:

The parallelogram and triangle laws are equivalent, and the circumstance may determine which one is used in a particular situation.

1 **Conceptual:** Are the parallelogram and triangle laws equivalent?

Answer: The two rules are equivalent since $\overrightarrow{\mathrm{AD}} - \overrightarrow{\mathrm{BC}}$.

2 **Factual:** Can you think of any situation where

a The parallelogram law cannot be used?

Answer: The parallelogram law cannot be used if the points are collinear.

b The triangle law cannot be used?

Answer: The triangle law can always be used.

3 How would you use these two laws to find $\overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{AD}}$?

Answer: Similar to real numbers, you can use the opposite vector for both laws.

4 Explain how you arrived at your result.

Answer: $\overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{AD}} = \overrightarrow{\mathrm{AB}} + (-\overrightarrow{\mathrm{AD}})$

5 In the diagram below, use each law separately to show that $\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} + \overrightarrow{\mathrm{CD}} + \overrightarrow{\mathrm{DE}} + \overrightarrow{\mathrm{EF}} = \overrightarrow{\mathrm{AF}}$.

Answer: When applying both laws, you can notice that the triangle law is much simpler to apply.

TOK

Analysis of the Page Rank formula provides a wonderful applied topic for linear algebra. You might research the "$25 billion dollar eigenvector".

Should mathematics be created for need or profit?

Investigation 4

This investigation has no TU. Its purpose is to highlight the properties of vector addition.

Vectors do satisfy the same properties as real numbers for the operation of addition. The only difference is that the identity element is the *zero vector* and the opposite element for each vector is the *opposite vector*.

Investigation 5

This investigation has no TU. Its purpose is to highlight the operation of multiplication of a vector by a scalar.

1 Draw diagrams illustrating **a** + **a** + **a** and **b** + **b** + **b** + **b** + **b** + **b**.

2 Comment on the magnitude and direction of the vector sums.

Answer: The vector **a** + **a** + **a** as a magnitude three times larger than a and the same direction as vector a, while vector **b** + **b** + **b** + **b** + **b** + **b** has a magnitude five times larger than **b** and the same direction as vector **b**.

3 Draw diagrams illustrating $-\mathbf{a} + (-\mathbf{a})$ and $-\mathbf{b} + (-\mathbf{b}) + (-\mathbf{b}) + (-\mathbf{b}) + (-\mathbf{b})$.

4 Comment on the magnitude and direction of these vector sums.

Answer: The vector $-\mathbf{a} + (-\mathbf{a})$ has a magnitude two times larger than **a** and the opposite direction to vector a, while vector $-\mathbf{b} + (-\mathbf{b}) + (-\mathbf{b}) + (-\mathbf{b}) + (-\mathbf{b})$ has a magnitude four times larger than **b** and the opposite direction to vector **b**.

5 Represent each vector addition in questions **1** and **3** as a single vector in terms of **a** and **b**.

Answer: $3\mathbf{a}$, $5\mathbf{b}$, $-2\mathbf{a}$, $-4\mathbf{b}$

6, 7 Use the diagrams to illustrate **i** $\frac{1}{2}\mathbf{a}$ **ii** $-\frac{5}{4}\mathbf{a}$ **iii** $\frac{4}{3}\mathbf{b}$ **iv** $-\frac{2}{5}\mathbf{b}$

and comment on the magnitude and direction of your vector results.

Answers: Vector $\frac{1}{2}\mathbf{a}$ has half the magnitude and the same direction as vector **a**. Vector $-\frac{5}{4}\mathbf{a}$ has $\frac{5}{4}$ the magnitude and the opposite direction to vector **a**. Vector $\frac{4}{3}\mathbf{b}$ has $\frac{4}{3}$ the magnitude and the same direction as vector **b**. Vector $-\frac{2}{5}\mathbf{b}$ has $\frac{2}{5}$ the magnitude and the opposite direction to vector **b**.

8 How can you represent the vector $-\mathbf{a} + (-\mathbf{a})$

i algebraically **ii** geometrically?

Answers: **i** a zero vector, 0

ii a point (both starting and end points are the same)

9.2 Introduction to vector algebra

Investigation 6

Factual: How do you obtain the components of the sum?

Answer: You obtain the components by adding the corresponding components.

TOK

The symbols used internationally for mathematics differ across countries and teachers.

Research and display some such as $y=$ and $f(x)$, the different ways of showing the domain, some using a comma as a decimal point, etc.

Which ones should you use? Why? Should we be exposed to all of the possibilities?

Investigation 7

Factual: How do you obtain the components of the vector $\lambda\mathbf{a}$?

Answer: Multiply each component in **a** by λ.

Investigation 8

Conceptual understanding:

Position and movement can be modelled in three-dimensional space using vectors.

1 What did you use to find the magnitude?

Answer: Pythagoras theorem.

2 **Factual:** What is the formula you used to find the magnitude?

Answer: The distance formula from a point A to the origin (0, 0).

3 How would you find the magnitude of a direction vector $\overrightarrow{AB}$?

Answer: By using the distance formula between two points A and B.

4 What did you use to find the magnitude?

Answer: Pythagoras theorem in a cuboid.

5 **Factual:** What is the formula you used to find the magnitude?

Answer: The distance formula from a point in A to the origin (0,0,0).

6 **Conceptual:** How can representing a vector in 3D enhance your knowledge of vectors?

Answer: Position and movement can be modelled in three-dimensional space using vectors.

Investigation 9

1 What can you conclude about the positions of the end points of your vectors?

Answer: End points will be scattered in all four quadrants but at an equal distance from the origin.

2 **Factual:** What shape do all the end points of the position vectors with the same magnitude form in two dimensions?

Answer: All the end points form a circle with the centre at the origin and a radius of 5 units.

3 Can you write an equation for the set of points?

Answer: $x^2 + y^2 = 25$

4 **Factual:** What shape do all the end points of position vectors with the same magnitude form in three dimensions?

Answer: All the end points in three dimensions form a sphere with the centre at the origin and the radius equal to the magnitude of one of the vectors.

9.3 Scalar product and its properties

Investigation 10

1 What do you notice?

Answer: The ratio of the product of magnitudes and the scalar product is always the same.

2 Which terms are proportional?

Answer: The scalar product is proportional to the magnitudes of vectors **a** and **b**.

3 How can you explain this proportionality?

Answer: This is because the angle between the vectors is always the same.

TOK

Here is an explanation from Peter Webb.

All laws for vector arithmetic must have a specific property, being that the answer to any vector arithmetic question must be independent of how you lay out the axis.

You can add two vectors by putting the tail of one on the head of the other and forming a triangle. I am sure you have done this many times. You don't need to lay out a set of axis to do this. Its purely geometric. If you want to do this algebraically, you can lay down some x-y axes and calculate $(a, b) + (c, d) = (a + c, b + d)$. But it doesn't matter how you lay down the axes, you get different numbers, but the resulting vector (the sum of the two vectors) is always the same.

There is an excellent reason why vector arithmetic has to be independent of the axis you select. Nature doesn't have axis, they are a human invention, so things in physics have to be independent of co-ordinate systems. And physics is believed to be isotropic - rotating an experiment should not affect an outcome, but it does affect the co-ordinate axis, so the layout of the coordinate system can't matter.

If you follow these rules, you have two choices about how vector multiplication in 2D space works. You can define multiplication as being a dot product which produces a scalar (a number), or the cross product which gives another vector perpendicular to the plane.

If you want vector arithmetic to model real physics, then the rules have to be independent of the co-ordinate system, and the dot and cross product are the only versions of multiplication that work, ie are independent of the (arbitrary) choice of axis. They are the only ones which are independent of the axis and hence can model real physics.

Investigation 11

2 How can you describe the curve?

Answer: The graph looks like a sine (cosine) curve.

3 Can you find an equation of the curve?

Answer: The curve has the amplitude of $\sqrt{5}$. $y = \sqrt{5}\cos(\theta)$.

Investigation 12

Conceptual understanding:

A positive scalar product results when the vectors form an acute angle between them. A negative scalar product results when the vectors form an obtuse angle between them. A scalar product of 0 implies the vectors are perpendicular, or have a right angle between them.

1 What can you say about the values of the scalar product?

Answer: The values of the scalar product are positive, negative or 0.

2 What can you say about the values of the angle?

Answer: Angles are always between $0 \text{ and } \pi$.

3 **Conceptual:** What is the relationship between the values of the scalar product and the angle between two vectors?

Answer: A positive scalar product results when the vectors form an acute angle between them. A negative scalar product results when the vectors form an obtuse angle between them. A scalar product of 0 implies the vectors are perpendicular, or have a right angle between them.

9.4 Vector equation of a line

Investigation 13

1 Take three distinctive lines in a plane and draw all possible mutual positions they can be in.

Answer: All three lines are parallel; two lines are parallel and one intersecting both of them; three pairs of lines intersecting at a point; all three lines intersecting at one point.

2 How many different positions can they form?

Answer: Four different positions.

3 Which position has a common point of all three lines?

Answer: Only one position with three concurrent lines.

TOK

A good class discussion as students might be more confident with previously encountered work.

9.5 Vector product and properties

Investigation 14

1 What do you notice?

Answer: The scalar products between vectors **a** and **c** and **b** and **c** are equal to 0.

2 What can you say about the vectors **a**, **b** and **c**?

Answer: The vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to both vectors **a** and **b**.

3 Is it true for other non-parallel vectors?

Answer: It is true for all parallel vectors **a** and **b**.

Investigation 15

Conceptual understanding:

The magnitude of the vector product has a proportional relationship to the magnitude of the vectors and the sine value of the angle between them.

1 What can you say about the ratio between the magnitude of the vector product and the product of the magnitudes?

Answer: The ratio is always positive and less than 1.

3 Can you describe the curve?

Answer: The graph looks like a sine wave.

4 **Conceptual:** What is the relationship between the magnitude of the vector product, the magnitudes of the vectors and the angle between the vectors?

Answer: The magnitude of the vector product has a proportional relationship to the magnitude of the vectors and the sine value of the angle between them.

Investigation 16

1 What do you notice about the results?

Answer: The first three columns are equal and the following three columns are opposite.

2 How can you justify these results?

Answer: The scalar product is commutative and the vector product is anti-commutative, therefore the opposite values.

9.6 Vector equation of a plane

Investigation 17

Conceptual understanding:

The solution for systems of equations can be carried out by a variety of equivalent algebraic and graphical methods.

1 Take three distinctive planes and draw all possible mutual positions they can have in a 3-dimensional space.

Answer: All three can be parallel, two parallel and one intersecting both of them, each two intersecting at a different line, all three intersecting at one line, all three intersecting at one point.

2 How many different positions can they form?

Answer: Five different positions.

3 How can you describe the sets of intersections?

Answer: No point of intersection, two lines of intersection, three lines of intersection (two planes at a time), one intersecting line of all three planes and one point of intersection of all three planes.

4 **Conceptual:** What is the relationship between the sets of intersections of three planes and the solutions of simultaneous equations?

Answer: Sets of intersections of three planes represent the solutions of systems of three linear equations with three unknowns, where unknowns represent the coordinates of the points of intersections. The solution for systems of equations can be carried out by a variety of equivalent algebraic and graphical methods.

TOK

A class discussion might include Vectors are used to solve many problems in position location. This can be to save a lost sailor or destroy a building with a laser guided bomb. Are either or both acceptable?

9.8 Application of vectors

TOK

Why are the symbolic representations of 3D shapes easier to work with than the actual drawings?

Three squares

Approaches to Learning/learner profile: Research, Critical Thinking

Exploration Criteria: Personal engagement (C), Use of mathematics (E)

IB Topic: Proof, Geometry, Trigonometry, Vectors

This is a good opportunity to revisit Proof that was introduced at the beginning of the book.

Proof is a difficult topic to pursue for an exploration as it will not always score highly in all criteria. Some suggestions are given at the end of how to possibly extend this and other topics which give the possibility of scoring better on Personal engagement (Criterion C) and even Reflection (Criterion D).

The problem

Some history

This problem first appeared in the "Mathematical Games" column of Scientific American in 1970. It was submitted by Martin Gardner who had been introduced to the problem from a friend, Lyber Katz, who had to do it for extra credit in Moscow when he was in grade 4. The problem was published again in the Journal of Recreational Math in 1971, where readers could try to solve it and send back their solutions. Many people, ranging from high school students to undergraduates and people in mathematics professions, sent back many different solutions. Charles W. Trigg gathered all of these solutions and published 54 different solutions under the title "A Three-Square Geometry Problem" in the following volume of the Journal of Recreational Math.

Emphasize to students that these 54 solutions, and others found since, lead to exactly the same answer. In mathematics class, students tend to focus on just finding the correct solutions rather than concentrating on **how** to find that answer. To find a solution one way may be easy, but to find another alternative solution to the same problem perhaps requires a deeper understanding of it, different skills and mathematical concepts. The idea of finding multiple solutions also links to the concept of divergent thinking, which is a crucial thought process used by mathematicians, engineers, architects and other problem solvers.

This is an interesting problem - not because the proofs of the answer are particularly difficult, but because there are so many of them!

Exploring the problem

This is an interesting crossover with TOK that looks at inductive and deductive reasoning and the concept of proof in mathematics and all other areas of knowledge.

The answer is 90° and students will likely get something close to this if they measure the angles or just guess.

Direct proof

$\alpha = 45°$

$\alpha = \beta + \phi = 45°$

$AC = \sqrt{2}$, $AD = \sqrt{5}$, $AE = \sqrt{10}$

To help students explain why the triangles are similar, ask:

- *Which sides correspond?*

SSS Similar triangle theorem

In ΔACD, the ratio of $CD : AC : AD = 1 : \sqrt{2} : \sqrt{5}$

In ΔACE, the ratio of $AC : CE : AE = \sqrt{2} : 2 : \sqrt{10} = 1(\sqrt{2}) : \sqrt{2}(\sqrt{2}) : \sqrt{5}(\sqrt{2})$

So $CD : AC : AD = AC : CE : AE$

Hence $\angle CAD = \angle CEA$ using properties of the two similar triangles.

Using exterior angle theorem:

$\angle ACB = \angle CAD + \angle ADC$

and so $\alpha = \beta + \phi$

and $\beta + \phi = 45°$

and so $\alpha + \beta + \phi = 90°$

Proof using an auxiliary line

$\angle BAC = \alpha$: Angle of isosceles right triangle.

$\angle EAF = \phi$: Alternate interior angles.

This will complete the proof because it will show that $\angle BAC + \angle EAF + \angle GAC = 90°$ since $\angle BAF$ is a right angle because it is an angle of a square.

To help students show that ΔGAC and ΔABD are similar, you could hint:

- *Calculate the lengths of sides CG and AC.*

$CG : AC = AB : BD$ since $\sqrt{2}/2 : \sqrt{2} = 1 : 2$

and $\angle GCA = \angle ABD = 90°$

$\angle GAC = \angle BDA = \beta$

These are equivalent angles in similar triangles.

Completing the proof:

$\angle BAC + \angle EAF + \angle GAC = 90°$

and so $\alpha + \beta + \phi = 90°$

Adding an auxiliary line is often a good method in a geometrical proof.

Proof using the cosine rule

$\angle XEY = \beta$ because it is a 2 by 1 triangle like ΔABD.

$AE = \sqrt{10}$ and $AY = \sqrt{5}$

$5^2 = (\sqrt{10})^2 + (\sqrt{5})^2 - 2(\sqrt{10})(\sqrt{5})\cos\theta$

$\cos\theta = -1/\sqrt{2}$

θ = 135°

β + ϕ = 180 - 135 = 45°

To complete the proof:

α = 45°

So α + β + ϕ = 90°

Proof using vectors

$\angle BEF = \beta$ because the square is 2 units long and 1 unit deep.

$\mathbf{a} = \binom{-3}{1}$ and $\mathbf{b} = \binom{-2}{-1}$**.0**

$$\cos(\mathrm{d}) = \frac{(-3)(-2)+(1)(-1)}{\sqrt{10}\sqrt{2}}$$

$$\cos(\mathrm{d}) = \frac{6-1}{\sqrt{10}\sqrt{5}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Therefore $d = \phi + \beta = 45°$

To complete the proof:

α = 45°

So α + β + ϕ = 90°

You could ask:

- *Which proof do you think is the* ***best****? Which do you* ***like*** *the most?*
- *What criteria are you using to make these judgements?*

Again, this is a good TOK point where the concept of 'best' can be discussed.

Students might consider best to be, for example:

- quickest or easiest
- most efficient or most elegant
- most surprising or using unusual mathematics
- hardest to understand.

Questions involving proof are a good starting point for an exploration, but would rarely score highly if that is all it involves as there is limited chance for personal engagement beyond possibly engaging in mathematics that is new to the student or beyond the scope of the course.

Extension

You could direct students towards problems in the textbook or problems on sites such as Nrich, Brilliant, or various mathematics competition sites.

Students could combine this with the work on Spearman's Rank and provide their classmates with different proofs and ask them to rank them.

The problems that students choose may also give them a chance to extend the problems and ask them to consider 'what if…?' to demonstrate Personal engagement (Criterion C).

For the problem in this task, they could discuss:

- *How else could the problem be extended?*
- *What about 4 squares, 5 squares, n squares?*

Students could then work on generalizing the problem.

This is good advice whenever a student tackles a problem where solutions are readily available by an internet search.

10 Equivalent systems of representation: more complex numbers

Essential understandings

Number and algebra allow us to represent patterns, show equivalences and make generalizations which enable us to model real-world situations. Algebra is an abstraction of numerical concepts and employs variables which allow us to solve mathematical problems.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Modelling real-life situations with the structure of arithmetic and geometric sequences and series allows for prediction, analysis and interpretation.
- Different representations of numbers enable equivalent quantities to be compared and used in calculations with ease to an appropriate degree of accuracy.
- Numbers and formulae can appear in different, but equivalent, forms, or representations, which can help us to establish identities.
- Formulae are a generalization made on the basis of specific examples, which can then be extended to new examples.
- Logarithm laws provide the means to find inverses of exponential functions which model real-life situations.
- Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns.
- The binomial theorem is a generalization which provides an efficient method for expanding binomial expressions.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
Complex numbers may be converted into different forms such as polar form which specifies the angle on the positive x-axis (argument) and modulus.	Investigation 1
The real part of a complex number represents the product of the modulus and cosine value of the argument while the imaginary part represents the product of the modulus and sine value of the argument.	Investigation 2
The multiple of the modulus of two complex numbers can be obtained by multiplying each modulus separately or multiplying the complex numbers first and then finding the modulus. The arguments of two complex numbers can be added by multiplying the complex numbers first and then finding the argument.	Investigation 3

The modulus of the reciprocal of a complex number can be represented as the reciprocal of the modulus of a complex number. The argument of the reciprocal of a complex number can be found by taking a full rotation and subtracting the argument of the complex number.	Investigation 4
Raising a complex number to any power results in the modulus raised to that power and the augment multiplied by that power.	Investigation 5
Solutions of the equation $z^n - 1 = 0, n \geq 3$ will form a regular polygon with *n* sides that has vertices equally spaced on the unit circle with one vertex at (1, 0). Representing complex numbers in the polar form and using the Argand diagram allows us to see the geometrical pattern and to solve seemingly difficult equation.	Investigation 6
The sum of an *n*th power and its reciprocal value is double the real part which is double the cosine of the *n*th multiple of the argument. The difference of an *n*th power and its reciprocal value is double the imaginary part multiplied by i, which is double the sine of the *n*th multiple of the argument multiplied by i.	Investigation 7

Syllabus sections covered in this chapter:

- AHL1.12
- AHL1.13
- AHL1.14

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 649: Equivalent systems of representation: more complex numbers	Page 658: Example 7 Page 664: Example 12 Page 668: Example 17 Page 672: Example 20	Page 652: Example 2 Page 654: Example 3 Page 670: Example 19	Pages 655, 663, 674

Assessment opportunities

End of chapter test	Chapter review	Exam practice
Page 675	Page 676	Page 677

10.1 Forms of a complex number

TOK

Links of Argand plane to Cartesian plane. Ethics: While Argand (1806) is generally credited with the discovery, the Argand diagram was actually described by Wessel before Argand. Caspar Wessel was a Danish–Norwegian mathematician and map maker. In 1799, seven years before Argand, Wessel described the geometrical interpretation of complex numbers as points in the complex plane.

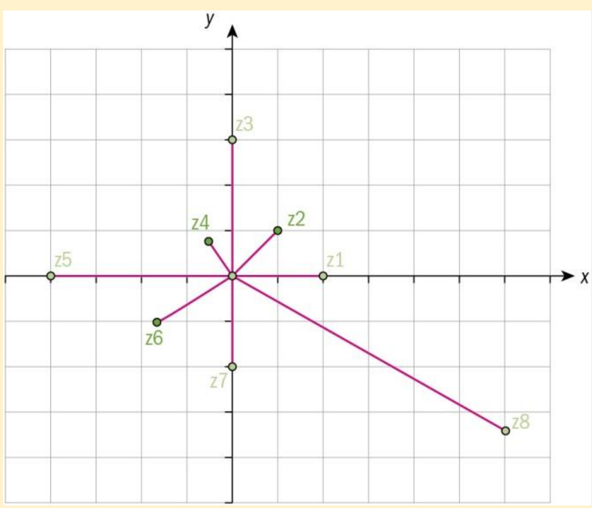
Investigation 1

Conceptual understanding:

Complex numbers may be converted into different forms such as polar form which specifies the angle on the positive *x*-axis (argument) and modulus.

1 Plot all of the complex numbers in the Argand plane.

Answer:



2 Draw a line segment that represents the modulus of each number and find its value.

Answer: $|z_1| = 2, |z_2| = \sqrt{2}, |z_3| = 3, |z_4| = 2, |z_5| = 4, |z_6| = 2, |z_7| = 2, |z_8| = 4\sqrt{3}$

3 Find the angle θ that the line segment encloses with the positive part of the *x*-axis for each complex number.

Answer: $\theta_1 = 0, \theta_2 = \frac{\pi}{4}, \theta_3 = \frac{\pi}{2}, \theta_4 = \frac{2\pi}{3}, \theta_5 = \pi, \theta_6 = \frac{7\pi}{6}, \theta_7 = \frac{3\pi}{2}, \theta_8 = \frac{11\pi}{6}$

4 **Conceptual:** Given a complex number $z = x + yi$, x, $y \in \Upsilon$, what is the relationship between the angle θ and the rectangular coordinates?

Answer: The angle is the inverse tangent value of the ratio of the imaginary and the real part of the complex number.

$$\begin{cases} \theta = \arctan\left(\frac{y}{x}\right), x > 0, y \geq 0 \\ \theta = \frac{\pi}{2}, x = 0, y \geq 0 \\ \theta = \pi + \arctan\left(\frac{y}{x}\right), x < 0, y \geq 0 \end{cases} \begin{cases} \theta = \pi + \arctan\left(\frac{y}{x}\right), x < 0, y \leq 0 \\ \theta = \frac{3\pi}{2}, x = 0, y \leq 0 \\ \theta = 2\pi + \arctan\left(\frac{y}{x}\right), x > 0, y < 0 \end{cases}$$

5 **Factual:** What is the value of the angle θ if the complex number is:

a real **b** imaginary?

Answer: **a** $\theta = 0$ or $\theta = \pi$ **b** $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

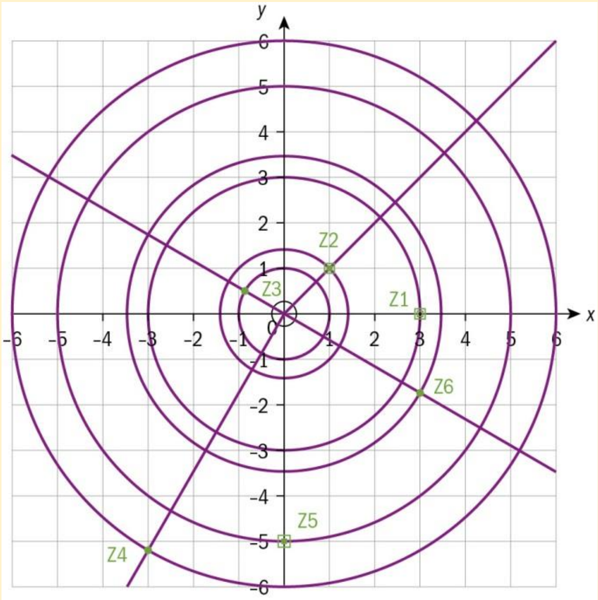
Investigation 2

Conceptual understanding:

The real part of a complex number represents the product of the modulus and cosine value of the argument while the imaginary part represents the product of the modulus and sine value of the argument.

1 Plot all of the complex numbers in the Argand plane given by the modulus and the argument.

Answer:



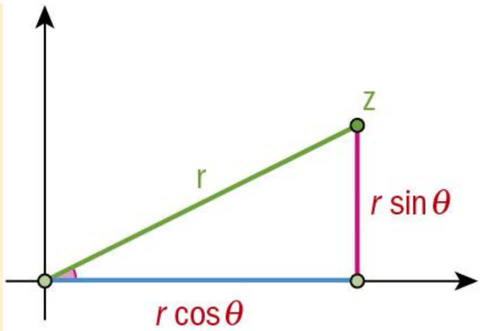
2 Find the rectangular coordinates, and therefore the Cartesian form, of all the complex numbers.

Answer: $z_1 = 3, z_2 = 1 + \mathrm{i}, z_3 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, z_4 = -3 - 3\sqrt{3}\mathrm{i}, z_5 = -5\mathrm{i}, z_6 = 3 - \sqrt{3}\mathrm{i}$

3 **Conceptual:** What is the relationship between the real and imaginary parts of a complex number and the modulus and argument of the same number?

Answer: The real part of a complex number represents the product of the modulus and cosine value of the argument while the imaginary part represents the product of the modulus and sine value of the argument.

$x = r\cos\theta, y = r\sin\theta$



TOK

Research for a definition of imagination and imaginary. Ask students to respond to the questions – “What are real numbers?”, “How can a number be imaginary?”

10.2 Operations with complex numbers in polar form

Investigation 3

Conceptual understandings:

The multiple of the modulus of two complex numbers can be obtained by multiplying each modulus separately or multiplying the complex numbers first and then finding the modulus.

The arguments of two complex numbers can be added by multiplying the complex numbers first and then finding the argument.

2 What is the relationship between the moduli of the factors and the modulus of the product?

Answer: $|z_1||z_2| = |z_1 z_2|$

3 What is the relationship between the arguments of the factors and the argument of the product?

Answer: $\arg(z_1) + \arg(z_2) = \arg(z_1 z_2) + 2k\pi, k = 0$ or 1

TOK

We can show a 2D or 3D coordinate in space and demonstrate a position in the classroom but what about an imaginary plane?

How can we represent an imaginary number in space? Does the imaginary plane exist? Where is it?

Is mathematics invented or was it already there to be found?

Investigation 4

Conceptual understandings:

The modulus of the reciprocal of a complex number can be represented as the reciprocal of the modulus of a complex number.

The argument of the reciprocal of a complex number can be found by taking a full rotation and subtracting the argument of the complex number.

2 What is the relationship between the modulus of the number and the modulus of the reciprocal number?

Answer: $\left|\frac{1}{z}\right| = \frac{1}{|z|}$

3 What is the relationship between the argument of the number and the argument of the reciprocal number?

Answer: $\arg\left(\frac{1}{z}\right) = 2\pi - \arg(z)$

TOK

The language of mathematics must be taken in context. If you were to look up words, such as complex, in a dictionary the meaning might well be confused. How do people around the world deal with this when these words are translated into a different language? What gets 'lost in translation' from one language to another?

10.3 Powers and roots of complex numbers in polar form

Investigation 5

Conceptual understanding:

Raising a complex number to any power results in the modulus raised to that power and the augment multiplied by that power.

1 Let $z = r \text{ cis } \theta$. Use the formula for multiplying two complex numbers in polar form to obtain expressions for: z^2, z^3, z^4 and z^5.

Answer: $z^2 = r^2\text{cis}(2\theta), z^3 = r^3\text{cis}(3\theta), z^4 = r^4\text{cis}(4\theta), z^5 = r^5\text{cis}(5\theta)$

2 Write the general expression for $n \in \mathbb{Z}^+$.

Answer: $z^n = r^n\text{cis}(n\theta), n \in \mathbb{Z}^+$

3 **Conceptual:** What is the relationship between the modulus and argument of the original number and the modulus and argument of its power?

Answer: The modulus of the power is power of the modulus and the argument of the power is the multiple of the argument.

4 **Factual:** Does the formula work for $n = 0$? Justify your answer.

Answer: The formula works for $n = 0$ because

$$z^0 = r^0\text{cis}(0 \cdot \theta) \Rightarrow 1 = 1\left(\underbrace{\cos 0}_{1} + \text{i}\underbrace{\sin 0}_{0}\right) \Rightarrow 1 = 1$$

5 **Factual:** Does the formula work for $n \in \mathbb{Z}^-$? Justify your answer.

Answer: From the previous section $\frac{1}{z} = \frac{1}{r}\text{cis}(-\theta) \Rightarrow z^{-1} = r^{-1}\text{cis}((-1)\cdot\theta)$ and then use the negative power rule: $z^{-n} = (z^n)^{-1} = \frac{1}{z^n} = \frac{1}{r^n\text{cis}(n\theta)} = r^{-n}\text{cis}((-n)\cdot\theta)$.

TOK

This is an opportunity to debate beauty and mathematics which can be followed up with the next TOK box about fractals. $e^{i\pi} + 1 = 0$ is beautiful in the eyes of mathematicians because it connects five of the most powerful values in mathematics in one equation.

TOK

This is an opportunity to explore fractals with the Mandelbrot and Julia sets.

Investigation 6

Conceptual understandings:

Solutions of the equation $z^n - 1 = 0, n \geq 3$ will form a regular polygon with n sides that has vertices equally spaced on the unit circle with one vertex at (1, 0).

Representing complex numbers in the polar form and using the Argand diagram allows us to see the geometrical pattern and to solve seemingly difficult equation.

1 Factorize the expressions and find exact forms of the solution to each equation.

Answer: $z^2 - 1 = 0 \Rightarrow (z-1)(z+1) = 0 \Rightarrow z = 1$ or $z = -1$

$z^3 - 1 = 0 \Rightarrow (z-1)(z^2+z+1) = 0 \Rightarrow z = 1$ or $z = \frac{-1 \pm \sqrt{3}i}{2}$

$z^4 - 1 = 0 \Rightarrow (z^2-1)(z^2+1) = 0 \Rightarrow z = \pm 1$ or $z = \pm i$

2 Write all the solutions in polar form and plot them on separate Argand diagrams.

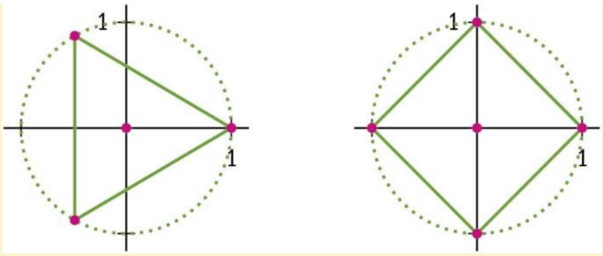
Answer: $n = 2: z = \text{cis}0$ or $z = \text{cis}\pi$

$n = 3: z = \text{cis}0$ or $z = \text{cis}\frac{2\pi}{3}$ or $z = \text{cis}\frac{4\pi}{3}$

$n = 4: z = \text{cis}0$ or $z = \text{cis}\frac{\pi}{2}$ or $z = \text{cis}\pi$ or $z = \text{cis}\frac{3\pi}{2}$

3 **Factual:** What do solutions of the second and third equation form on the diagrams?

Answer:



Solutions of the second equation form an equilateral triangle that has vertices equally spaced on the unit circle with one vertex at (1, 0). Solutions of the third equation form a square that has vertices equally spaced on the unit circle with one vertex at (1, 0).

4 Can you predict the shape and write the polar form for the solutions of the equation $z^5-1 = 0$?

Answer: Solutions of the equation $z^5-1=0$ will form a regular pentagon that has vertices equally spaced on the unit circle with one vertex at (1, 0).

$$n=5: z=\text{cis}0 \text{ or } z=\text{cis}\frac{2\pi}{5} \text{ or } z=\text{cis}\frac{4\pi}{5} \text{ or } z=\text{cis}\frac{6\pi}{5} \text{ or } z=\text{cis}\frac{8\pi}{5}$$

5 **Conceptual:** What do solutions of the equation $zn-1=0$, $n\geq3$ form on the Argand diagram?

Answer: Solutions of the equation $z^n-1=0, n\geq3$ will form a regular polygon with n sides that has vertices equally spaced on the unit circle with one vertex at (1, 0).

6 Use the geometrical representation of the solutions to predict the polar form of the solutions of the equation $z^n - 1 = 0$, $n \geq 3$.

Answer:

$$n: z=\text{cis}0 \text{ or } z=\text{cis}\frac{2\pi}{n} \text{ or } z=\text{cis}\frac{4\pi}{n} \text{ or } z=\text{cis}\frac{6\pi}{n} \ldots, z=\text{cis}\frac{2k\pi}{n}, k=0,1,\ldots n-1$$

7 **Conceptual:** How does representing solutions in a different form allow you to solve $z^n - 1=0$, $n \geq 3$?

Answer: Representing complex numbers in the polar form and using the Argand diagram allows us to see the geometrical pattern and to solve seemingly difficult equation.

TOK

Consider i as an imaginary number, there is no number equivalent to the square root of negative one, yet when we square it we get a real number. Can things that are imaginary become real?

In Physics they often simplify calculations. One example is: the voltages and currents in an electronic circuit have real values, but in AC problems, where they change sinusoidally with time, we can represent them as complex numbers and thus include the amplitude and phase of the variation in one number. You can then perform calculations, in arithmetic form. with these numbers to work out what is going on in the circuit, which is simpler than having to solve coupled differential equations to get the form of the functions.

The depiction of a complex number as a point in the plane was significant because it made the idea of complex numbers more acceptable. In particular, this visualization helped "imaginary" and "complex" numbers become accepted in mainstream mathematics as a natural extension to negative numbers along the real line.

TOK

You can start a debate using: Was "i" there all of the time or was it invented by mathematicians? Have one team talk for and one against the "invention", and have a third team rule on the debate with reasons.

Investigation 7

Conceptual understanding:

The sum of an *n*th power and its reciprocal value is double the real part which is double the cosine of the *n*th multiple of the argument. The difference of an *n*th power and its reciprocal value is double the imaginary part multiplied by i, which is double the sine of the *n*th multiple of the argument multiplied by i.

1 Use De Moivre's theorem to find the following sums $z+\frac{1}{z}, z^2+\frac{1}{z^2}, z^3+\frac{1}{z^3}$ and $z^4+\frac{1}{z^4}$ in their simplest form.

Answer: $z^2+\frac{1}{z^2}=2\cos 2\theta, z^3+\frac{1}{z^3}=2\cos 3\theta,\ z^4+\frac{1}{z^4}=2\cos 4\theta$

2 **Factual:** What can you say about the real and the imaginary parts of these sums?

Answer: The imaginary parts are all equal to 0, therefore the sums are real numbers.

3 **Factual:** What is the general formula for $z^n+\frac{1}{z^n}, n\in\mathbb{Z}^+$?

Answer: $z^n+\frac{1}{z^n}=2\cos n\theta$

4 Use De Moivre's theorem to find the following differences $z-\frac{1}{z}, z^2-\frac{1}{z^2}, z^3-\frac{1}{z^3}$ and $z^4-\frac{1}{z^4}$ in their simplest form.

Answer: $z-\frac{1}{z}=2\mathrm{i}\sin\theta, z^2-\frac{1}{z^2}=2\mathrm{i}\sin 2\theta, z^3-\frac{1}{z^3}=2\mathrm{i}\sin 3\theta$ and $z^4-\frac{1}{z^4}=2\mathrm{i}\sin 4\theta$

5 **Factual:** What can you say about the real and the imaginary parts of these differences?

Answer: The real parts are all equal to 0, therefore the differences are purely imaginary numbers.

6 **Factual:** What is the general formula for $z^n-\frac{1}{z^n}, n\in\mathbb{Z}^+$?

Answer: $z^n-\frac{1}{z^n}=2\mathrm{i}\ \sin n\theta$

7 **Conceptual:** What is the relationship between the expressions $z^n+\frac{1}{z^n}$ and $z^n-\frac{1}{z^n}, n\in\mathbb{Z}^+$ in polar form?

Answer: The sum of an *n*th power and its reciprocal value is double the real part which is double the cosine of the *n*th multiple of the argument. The difference of an *n*th power and its reciprocal value is double the imaginary part multiplied by i, which is double the sine of the *n*th multiple of the argument multiplied by i.

The cubic formula

Approaches to Learning: Critical thinking, transfer skills

Exploration Criteria: Mathematical communication (B), Use of mathematics (E)

IB Topic: Polynomials, Complex numbers, roots, Algebraic manipulation

Introduction

This RLT follows up on the comment made in the chapter regarding the origin of complex numbers coming from the work of sixteenth century mathematicians on solving cubic equations. Some of the steps in the derivation require some quite tricky algebra but this is a good opportunity for you to highlight and for students to demonstrate rigor in their approach to the topic which in a requirement for high levels in Criterion E.

The use of a worked example alongside the general approach is also a good aspect to point out to the students as it can help to clarify the steps they are taking in their own exploration if it follows a similar theme.

Solving a quadratic equation

The quadratic equation has solutions 27 and −1.

Solving a cubic equation

STEP **1**

ALGEBRA CHALLENGE 1:

Making a substitution $x = y - \frac{b}{3a}$:

$$a\left(y - \frac{b}{3a}\right)^3 + b\left(y + \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0$$

Simplifying:

$$ay^3 + \left(c - \frac{b^2}{3a}\right)y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0$$

Rewriting in the required form:

$$y^3 + \frac{1}{a}\left(c - \frac{b^2}{3a}\right)y + \frac{1}{a}\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0$$

Where $\mathrm{e} = \frac{1}{a}\left(c - \frac{b^2}{3a}\right)$ and $f = \frac{1}{a}\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)$.

Worked example:

$a = 1$, $b = -9$, $c = 36$, $d = -80$

Therefore $e = 9$ and $f = -26$

The fact that the depressed cubic form of the equation $x^3 - 9x^2 + 36x - 80 = 0$ is $y^3 + 9y - 26 = 0$

follows from above.

STEP 2

ALGEBRA CHALLENGE 2:

Substitute $y = z - \frac{e}{3z}$ into $y^3 + 9y - 26 = 0$.

Expand, muliply all terms by z^3 and simplify to get $z^6 + fz^3 - \frac{e^3}{27} = 0$.

STEP 3

Worked example:

Following from above:

$$w^2 + fw - \frac{e^3}{27} = 0$$
$$w^2 - 26w - \frac{9^3}{27} = 0$$
$$w^2 - 26w - 27 = 0$$

Note: Once the values of *e* and *f* are known it is possible to skip STEP 2 and move directly to the reduced quadratic form.

From the start, the two roots are 27 and −1.

Undoing the substitutions

Now from these values of *w* students will 'undo' the subsitutions to find the values of *x* that solve the original cubic.

Students can use the method of finding roots of an equation introduced in Section 10.3 of this chapter for this.

Solve for either w = 27 or w = −1.

They will both produce the same answers

This is the method for w = 27:

$$z_1 = (27)^{\frac{1}{3}}(\cos\frac{0}{3} + \mathrm{i}\sin\frac{0}{3})$$
$$= 3$$
$$z_2 = (27)^{\frac{1}{3}}(\cos\frac{0+2\pi}{3} + \mathrm{i}\sin\frac{0+2\pi}{3})$$
$$= 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{3}\right)$$
$$= -\frac{3}{2} + i\frac{3\sqrt{3}}{2}$$

$$z_3 = (27)^{\frac{1}{3}}\left(\cos\frac{0+4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 3\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 3\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{3}{2} - i\frac{3\sqrt{3}}{2}$$

Now since

$$y = z - \frac{3}{z}$$

$$y_1 = z_1 - \frac{3}{z_1}$$

$$= 3 - \frac{3}{3}$$

$$= 2$$

$$y_2 = z_2 - \frac{3}{z_2}$$

$$= \left(-\frac{3}{2} + i\frac{3\sqrt{3}}{2}\right) - \frac{3}{\left(-\frac{3}{2} + i\frac{3\sqrt{3}}{2}\right)}$$

$$= -\frac{5+3i\sqrt{3}}{-1+i\sqrt{3}}$$

$$= -\frac{5+3i\sqrt{3}}{-1+i\sqrt{3}} \times \frac{-1-i\sqrt{3}}{-1-i\sqrt{3}}$$

$$= -1 + i2\sqrt{3}$$

$$y_3 = z_3 - \frac{3}{z_3}$$

$$= \left(-\frac{3}{2} - i\frac{3\sqrt{3}}{2}\right) - \frac{3}{\left(-\frac{3}{2} - i\frac{3\sqrt{3}}{2}\right)}$$

$$= \frac{5-i3\sqrt{3}}{1+i\sqrt{3}}$$

$$= \frac{5-i3\sqrt{3}}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$

$$= -1 - i2\sqrt{3}$$

Since

$$\begin{aligned} x &= y + 3 \\ x_1 &= y_1 + 3 \\ &= 2 + 3 \\ &= 5 \\ x_2 &= y_2 + 3 \\ &= (-1 + i2\sqrt{3}) + 3 \\ &= 2 + i2\sqrt{3} \\ x_3 &= y_3 + 3 \\ &= (-1 - i2\sqrt{3}) + 3 \\ &= 2 - i2\sqrt{3} \end{aligned}$$

The solutions of the cubic are

x_1, x_2, and x_3, that is,

$$5, 2 + i2\sqrt{3}, 2 - i2\sqrt{3}$$

It can be shown that using $w = -1$ would give the same values as $w = 27$.

Students could try this as **extension** work.

Extension

For the Del Ferros general formula:

Students could test the formula on the example to obtain the real solution of $x = 5$.

From this it would be possible to determine a factor of $x - 5$.

This can be divided into the cubic to obtain a quadratic, which can be solved to obtain the two complex roots.

For the investigation of the discriminant of the cubic formula:

The discriminant is

$$\frac{p^2}{4} + \frac{p^3}{27}$$

The equation with have:

three real solutions at least two of which are equal if $\frac{p^2}{4} + \frac{p^3}{27} = 0$

one real root and two conjugate imaginary roots if $\frac{p^2}{4} + \frac{p^3}{27} > 0$

three distinct real roots if $\frac{p^2}{4} + \frac{p^3}{27} < 0$.

11 Valid comparisons and informed decisions: probability distributions

Essential understandings

Probability enables us to quantity the likelihood of events occurring and so evaluate risk. Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned, in detail, to differentiate between the theoretical and the empirical/observed. Probability theory allows us to make informed choices, to evaluate risk, and to make predictions about seemingly random events.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Organizing, representing, analysing and interpreting data and utilizing different statistical tools facilitates prediction and drawing of conclusions.
- Different statistical techniques require justification and the identification of their limitations and validity.
- Approximation in data can approach the truth but may not always achieve it.
- Some techniques of statistical analysis, such as regression, standardization or formulae, can be applied in a practical context to apply to general cases.
- Modelling through statistics can be reliable, but may have limitations.
- Properties of probability density functions can be used to identify measure of central tendency such as mean, mode and median.
- Probability methods such as Bayes theorem can be applied to real-world systems, such as medical studies or economics, to inform decisions and to better understand outcomes.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
The probability of multiple events occurring can be evaluated by finding the probability of each event separately without double counting the intersections.	Investigation 1
Bayes' theorem describes the probability of an event happening, considering the possible conditional constraints.	Investigation 2
Creating a probability model for a real-life event can be reliable, but may have limitations.	Investigation 4
Representing outcomes in a sample space facilitates finding the	Investigation 5

probability distribution function.	
By understanding the effects of linear functions on discrete random variables, applications to general cases can be made.	Investigation 6
Considering Pascal's triangles leads to generalising a formula for the binomial distribution.	Investigation 7
The definition of expectation can be extended to the binomial distribution.	Investigation 8
By visually representing the normal distribution the effects of changing the parameters μ and σ^2 can be observed.	Investigation 9

Syllabus sections covered in this chapter:

- SL4.2*
- SL4.5*
- SL4.6*
- SL4.7*
- SL4.8*
- SL4.9*
- SL4.11
- SL4.12
- AHL4.13
- AHL4.14
- AHL4.10

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 681: Valid comparisons and informed decisions: probability distributions	Page 698: Example 7 Page 709: Example 12	Page 714: Example 14 Page 715: Example 15 Page 721: Example 18 Page 722: Example 19	Pages 695, 705, 711, 717, 724

Assessment opportunities

End of chapter test	Chapter review	Exam practice
Page 724	Page 727	Page 728

11.1 Axiomatic probability systems

TOK

Background research on "Paul the octopus" and "groundhog day" where Punxsutawney Phil is a groundhog, immortalised in the film *Groundhog Day*, can help produce interesting discussions.

Do you believe that people can predict the future with certainty? Can you give examples? If so, why do you believe this?

Which of the following are involved in prediction and how? Reason, intuition, emotion, faith.

Investigation 1

Conceptual understanding:

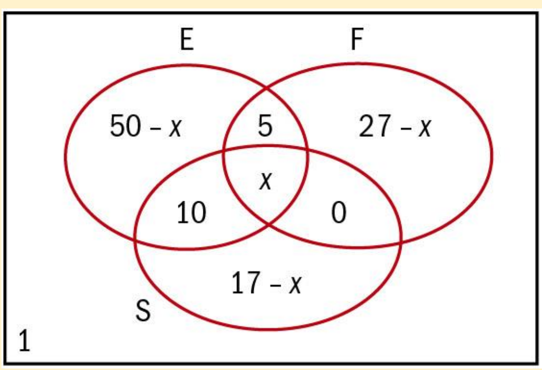
The probability of multiple events occurring can be evaluated by finding the probability of each event separately without double counting the intersections.

1 Represent this information in a Venn diagram.

Answer: A Venn diagram should always consist of the universal set *U*, the rectangle and in this case three overlapping ovals to represent the sets *E, F* and *S.*

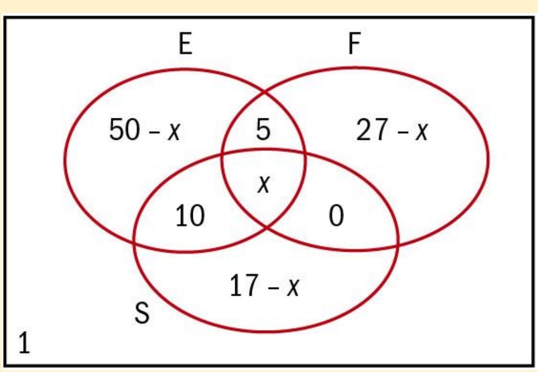
$U = 100$

The information given ensures the insertion of 5,1,10,0.



2 Find the number of factory workers who speak all three languages.

Answer: Let x be the number of workers who speak all three languages, use this information and the values for *E*, *F* and *S* to obtain this Venn diagram:

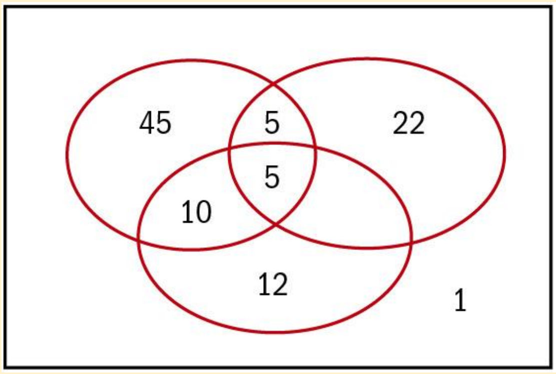


$$(50-x)+10+5+(27-x)+x+(17-x)+1=100$$

$$110-2x=100$$

$$x=5$$

Therefore the data can be represented in a Venn diagram ready for analysis and interpretation.



3 Use the numbers given in the Venn diagram to calculate the following probabilities.

a P($F \cap S \cap E$) **b** P(F') **c** P($E \cup F$) **d** P($E \cup F'$)

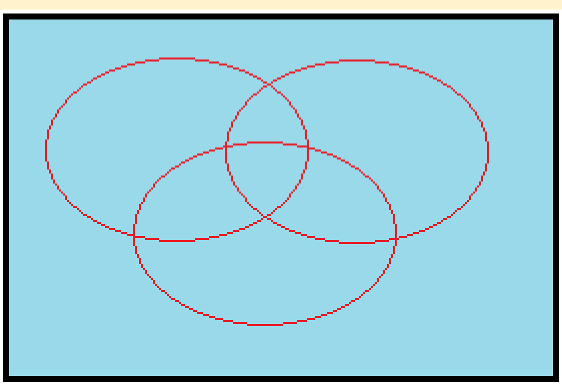
Answers: **a** **a** $\frac{5}{100}=\frac{1}{20}$ **b** $\frac{68}{100}=\frac{17}{25}$ **c** $\frac{87}{100}$ **d** $\frac{78}{100}=\frac{39}{50}$

4 Use the Venn diagram to show the following results.

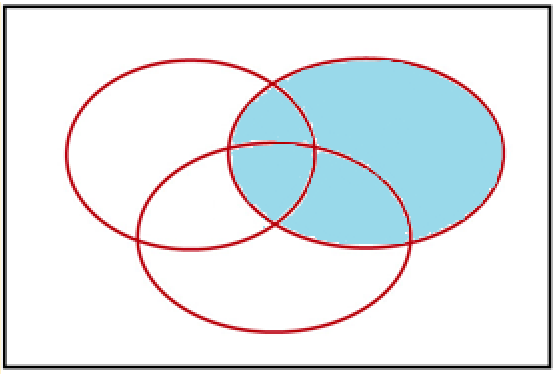
a P($E \cup U$) = P(U) **b** P($F \cap U$) = P(F) **c** P($F \cap \emptyset$) = P($\emptyset$) **d** P($S \cup \emptyset$) = P(S)

e P($E \cup E'$) = P(U) **f** P($E \cap E'$) = P($\emptyset$) **g** P(U) = 1

Answers:



a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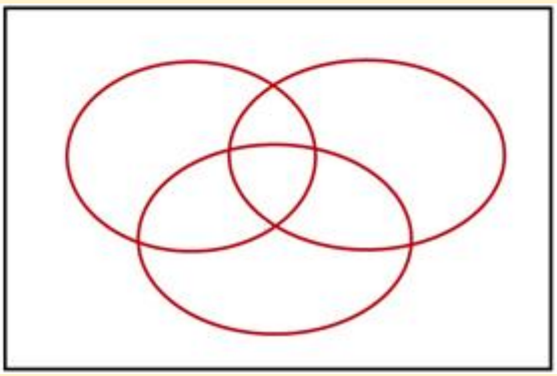
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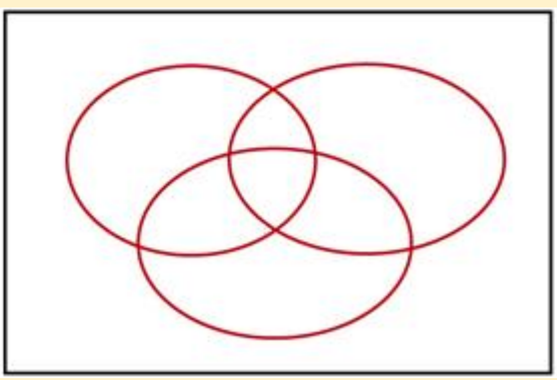
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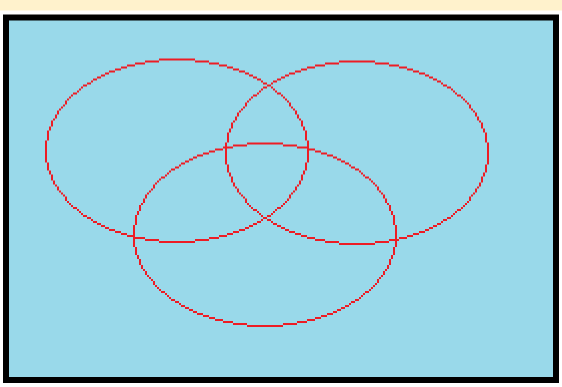
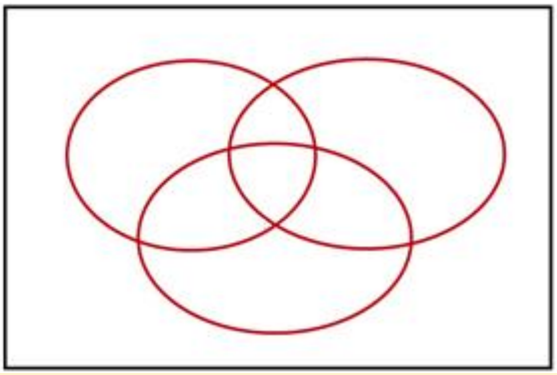
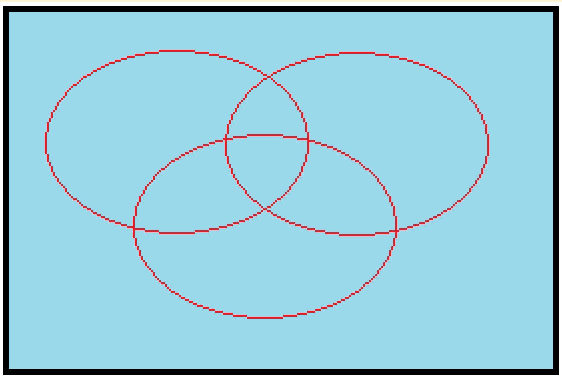

e



f



g

5 **Conceptual:** How do the results from **3c** relate to the individual probabilities P(E) and P(F)?

Answer: The Venn diagram allows you to organise, represent and interpret the data to see that $\mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F)$

6 **Conceptual:** In words, how do the results from **3c** relate the individual probabilities P(E) and P(F)? Why do you subtract the intersection when finding the probability of E and F?

Answer: The probability of multiple events occurring can be evaluated by finding the probability of each event separately without double counting the intersections.

Investigation 2

Conceptual understanding:

Bayes' theorem describes the probability of an event happening, considering the possible conditional constraints.

1 Find P(A) and P(A').

Answer: $P(A) = \dfrac{2}{52} = \dfrac{1}{26}; P(A') = \dfrac{50}{52} = \dfrac{25}{26}$

2 Copy and complete the tree diagram.

Answer:

$$\left.\begin{aligned} P(B|A) &= \frac{25}{51} \\ P(B'|A) &= \frac{26}{51} \end{aligned}\right\}$$ since a black ace was drawn, we are left with 25 black cards and 26 red cards.

$$\left.\begin{aligned} P(B'|A') &= \frac{25}{51} \\ P(B|A') &= \frac{26}{51} \end{aligned}\right\}$$ since a red ace was drawn, we are left with 26 black cards and 25 red cards.

3 What do the four outcomes on the far right represent? (Explain your answer)

Answer:

$P(A \cap B)$ the probability a black ace was drawn followed by a black card

$P(A \cap B')$ the probability a black ace was drawn followed by a red card

$P(A' \cap B)$ the probability a red ace was drawn followed by a black card

$P(A' \cap B')$ the probability a red ace was drawn followed by a red card

4 How would you calculate the outcomes on the far right?

Answer: To calculate the outcomes on the far right you would have to multiply along the branches, because these events are now independent of each other.

5 How can you verify the answer you obtain when you add the values for the outcomes on the far right?

Answer: When you add the outcomes on the far right you would obtain 1, because they represent all the possibly outcomes of drawing two cards successively.

TOK

Intuition is often not a good way of knowing in Probability. There are lots of other examples in this field where intuition potentially lets you down. What about in other areas of knowledge? Is it more or less reliable there?

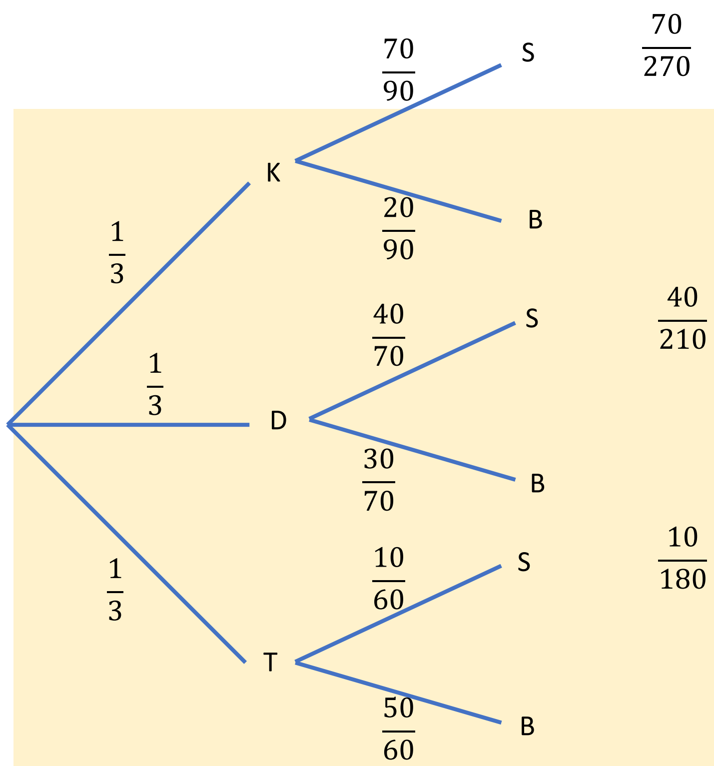
Investigation 3

1 If all the boundaries were ignored, find the probability their sister chooses a strawberry plant.

Answer: $P(S) = \dfrac{n(S)}{n(U)} = \dfrac{120}{220} = \dfrac{6}{11}$

2 Draw a tree diagram to show the possibilities if she picks one of the plots at random first.

Answer:



3 Find the probability that she picks a strawberry plant given that she has picked a plot at random.

Answer:
$$P(S)=\frac{70}{270}+\frac{40}{210}+\frac{10}{180}$$
$$=\frac{191}{378}$$

4 Why are the answers for questions **1** and **3** not equal?

Answer: The answers for part 1 and part 3 are not the same because the condition "she picks a plot at random" has been given and therefore you must take into consideration this extra constraint.

5 Given that she chose a strawberry plant, find the probability that it came from Kaita's area.

Answer: $\mathrm{P}(K\,|\,S)=\frac{\mathrm{P}(K\cap S)}{\mathrm{P}(S)},\ \frac{\frac{70}{270}}{\frac{191}{378}}=\frac{98}{191}$

This leads into Bayes theorem, to inform decisions and to better understand outcomes. Bayes' theorem describes the probability of an event happening, considering the possible conditional constraints.

11.2 Probability distributions

Investigation 4

Conceptual understanding:

Creating a probability model for a real-life event can be reliable, but may have limitations.

1,2 Answers will vary.

3 Conceptual: How does this situation mimic the real-life situation?

Answer: This will produce a random data set which will mimic a classroom situation, as there are 10 digits sorting them into 1–9 and 0 will generate simulated sets where P(1–9) = 0.9 and P(0) = 0.1.

4,5 Answers will vary.

6 What are the values the variable can take?

Answer: y-value from 0 to 25 but the probability of getting more than 7 is very small so you can write greater than 7 as a value.

7 What relation does it have to the variable the art teacher is really interested in?

Answer: This allows the art teacher to see the distribution of the number of left-handed students in his classes.

8 Answers will vary.

9 From the data determine how many pairs of left-handed scissors he should buy.

Answer: The Art teacher should buy 6 sets of scissors for a class of 25 students as the probability of more than 6 left-handed students is very low.

Investigation 5

Conceptual understanding:

Representing outcomes in a sample space facilitates finding the probability distribution function.

1 Draw a sample space for the outcomes.

Answer:

2	3	4	5
3	4	5	6
4	5	6	7
5	6	7	8

2 Construct a probability distribution table to represent the associated probabilities.

Answer:

	2	3	4	5	6	7	8
	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

3 Find the probability distribution function.

Answer: When $2 < x < 5, f(x) = \frac{x-1}{16}$; when $6 < x < 8, f(x) = \frac{9-x}{16}$

4 Verify that X is a random variable.

Answer: Sum of probabilities equals 1.

5 Factual: Is there more than one probability distribution that X can take?

Answer: No, there is only one probability distribution that X can take.

6 Conceptual: Why is representing outcome on a sample space useful?

Answer: Representing outcomes in a sample space facilitates finding the probability distribution function.

Investigation 6

Conceptual understanding:

By understanding the effects of linear functions on discrete random variables, applications to general cases can be made.

1 Write down an expression for Y in terms of X and 3.

Answer: $Y = X + 3$

2 Will the associated probabilities change?

Answer: No

3 Copy and complete a probability distribution table of X and Y.

Answer:

x	0	1	2	3	4
y	3	4	5	6	7
P	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{5}{24}$	$\frac{1}{8}$	$\frac{1}{4}$

4 Find E(X).

Answer: $(\frac{1}{4})+(\frac{10}{24})+(\frac{6}{8})+1=20.416666$

5 Find E(Y).

Answer: 23.41666

6 Generalise a result for E($R + c$) where R is a random variable and c is a constant.

Answer: E(R) + c. This leads to understanding the effects of linear functions to apply to general cases.

7 **Conceptual:** What are the effects of linear functions on random variables?

Answer: If you add or subtract a constant to the random variable X, a new random variable Y can be formed. The mean of this new random variable Y equals the mean of the previous variable X plus or minus the constant.

8 **Conceptual:** Why is it useful to look at the effects of linear functions on discrete random variables?

Answer: By understanding the effects of linear functions on discrete random variables, applications to general cases can be made.

TOK

A class discussion might go like this:

Social scientists use statistics to study human behaviour, for example, population, income, birth rates. Can you think of some more examples of this?

The United Nations and World Health Organization collect data and use it to help plan aid and development programs.

Can you think of some aspects of human behaviour from other areas of knowledge that cannot be measured?

TOK

This can lead to a debate about what is a game and what is mathematics.

What is mathematics?

"I think many physicists, including myself, agree that there should be some complete description of the universe and the laws of nature. Implicit in that assumption is the universe is intrinsically mathematical." – Simeon Hellerman

What is Mathematics? “Very simple: Mathematics is the science of structure, order, and relation that has evolved from elemental practices of counting, measuring, and describing the shapes of objects. It deals with logical reasoning and quantitative calculation, and its development has involved an increasing degree of idealization and abstraction of its subject matter. Since the 17th century, mathematics has been an indispensable adjunct to the physical sciences and technology, and in more recent times it has assumed a similar role in the quantitative aspects of the life sciences”. - Jorge Morales Pedraza

TOK

A good question for a debate or blog post would be - Ethics is an area of knowledge in its own right. Where do you see an intersection of the AOKs of mathematics and ethics?

When taking a chance decision in your life, which skills do you rely and in which order? Intuition (your gut feeling)? Reason? Emotion? Memory? Faith? Imagination?

Developing Inquiry skills

Consider now the first option from the original problem.

What is the best strategy?

Pay \$10 and Roll 1 dice

If you get a 6 win \$30

If you get a 5 or 4 get your money back

If you get a 1, 2 or 3 you get nothing!

Is this a game that you think that if you played a few times you would end up with more than you started with?

Answer: Students should make a guess based on intuition for the problem

How could we tell?

Answer: Calculate the expected value

Define a random variable for the winnings from playing the game once.

Answer: Let W = amount of winnings (note that this is not the score on the dice)

Draw a probability distribution for your random variable.

Answer:

w	20	0	-10
P(W=w)	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

Calculate the expected winnings from the game.

Answer: $E(W)=\left(20\times\frac{1}{6}\right)+\left(0\times\frac{2}{6}\right)+\left(-10\times\frac{3}{6}\right)=-\frac{5}{3}$

Expected winnings are therefore -\$1.67 (there is an expected loss of \$1.67 each time the game is played)

How much would you expect to win or lose if you played the game 10 times? 100 times?

For 10 times expected loss is \$16.67 and for 100 times it is \$166.67

Is the game 'fair'?

No, because you expect to lose money.

How could you define a fair game?

A game where the expected winnings are 0. (E(W)=0)

What adjustment could you make to the prizes that would ensure that this is a fair game?

We want E(W) = 0

So, for example, you could increase the winnings when rolling a 6 to £x, where x is such that E(W) = 0.

$$E(W)=\left(x\times\frac{1}{6}\right)+\left(0\times\frac{2}{6}\right)+\left(-10\times\frac{3}{6}\right)=0$$

$$\frac{x}{6}-5=0$$

$$\frac{x}{6}=5$$

$$x=30$$

The first prize could be changed to \$40.

Other solutions and changes are possible too.

11.3 Continuous random variables

TOK

You might want to use this TOK session to begin a discussion about whether all people possess the same reasoning ability at the same level. Some might argue that our ability to reason distinguishes us from the rest of the animal kingdom. You might want to ask where students have used reason today. As a counterclaim you could point to people who have a diminished capacity for reasoning, such as the mentally disabled, and ask if they are not human. An attention-grabbing debate is sure to follow!

11.4 Binomial distribution

Investigation 7

Conceptual understanding:

Considering Pascal's triangles leads to generalising a formula for the binomial distribution.

1 State the number of trials n.

Answer: 5

2 State the outcomes.

Answer: Obtaining a six or not obtaining a 6.

3 Calculate the probability of success in each trial.

Answer: $\frac{1}{6}$

4 Define the random variable and write all of its values.

Answer: 0,1,2,3,4,5

5 Deduce a formula for the number of variates in a binomial distribution.

Answer: $x + 1$

6 One possible outcome is SFSSF, where S indicates success and F failure. Calculate the probability of this outcome.

Answer: $(\frac{1}{6})^3(\frac{5}{6})^2$

7 **Factual:** How many outcomes are there where there are three successes and two failures?

Answer: 0.003215

8 Calculate the probability of three successes and two failures.

Answer: 10

9 If p is the probability of success and $(1 - p)$ is the probability of failure, find the probability of 0, 1, 2, 3, 4, 5 successes in five independent trials.

Answer:

$$P(X=0)=1(1-p)^5p^0$$
$$P(X=1)=5(1-p)^4p^1$$
$$P(X=2)=10(1-p)^3p^2$$
$$P(X=3)=10(1-p)^2p^3$$
$$P(X=4)=5(1-p)^1p^4$$
$$P(X=5)=1(1-p)^0p^5$$

10 Hence, generalize a formula for $P(X = r)$ in a binomial distribution with n fixed trials.

Answer: $P(X=x)=\binom{n}{x}px \quad (1-p)^{n-x},\ x=0,1,2,3,\ldots,n$

11 **Conceptual:** How are Pascal's triangle and binomial theorem related?

Answer: This leads to an understanding of Probability methods such as the binomial distribution can be applied to real-world systems, to inform decisions and to better understand outcomes and generalising a formula for the binomial distribution.

TOK

We often use a theoretical distribution, such as the binomial theorem, to describe a random variable that occurs in a real-life situation. This process is called modelling and enables us to make calculations and possibly predict.

However, it is unusual for a model to fit a real-life situation perfectly and we have to be ready for some error, which is a normal situation in life. However, we can often give a percentage chance of an event like forecasting rain or the success of a medical procedure.

Investigation 8

Conceptual understanding:

The definition of expectation can be extended to the binomial distribution.

1 Copy and complete the probability distribution table.

Answer:

x	0	1	2	3	...	n
P(X=x)	$\binom{n}{0}p0\ \ (1-p)^n$	$\binom{n}{1}p1\ \ (1-p)^{n-1}$	$\binom{n}{2}p^2(1-p)^{n-2}$	$\binom{n}{3}p^3(1-p)^{n-3}$		$\binom{n}{n}p^n(1-p)^{n-n}$
P(X=x)	$(1-p)^n$	$np(1-p)^{n-1}$	${}^nC_2p^2(1-p)^{n-2}$	${}^nC_3p^3(1-p)^{n-3}$		${}^nC_np^n(1-p)^{n-n}$ $= p^n$

2 Write a sum for E(X).

Answer: e.g. $\mathrm{E}(X) = 0\times(1-p)^n + 1\times np(1-p)^{n-1} + \ldots\ldots + n\times p^n$, $\mathrm{E}(X) = \sum x\mathrm{P}(X=x)$

3 **Factual:** Which factors are present in each term?

Answer: n and p

4 Use the definition for polynomials to simplify the expression.

Answer:

$$(0)(1-p)^n + (1)np(1-p)^{n-1} + \frac{(2)n(n-1)}{2!}p^2(1-p)^{n-2} + \frac{(3)n(n-1)(n-2)}{3!}p^3(1-p)^{n-3} + \ldots + np^n$$
$$= 0 + np\left[(1-p)^{n-1} + \frac{(2)(n-1)}{2!}p(1-p)^{n-2} + \frac{(3)(n-1)(n-2)}{3!}p^2(1-p)^{n-3} + \ldots + p^{n-1}\right]$$
$$= np[(1-p+p)^{n-1}]$$
$$= np$$

5 **Conceptual:** How can we calculate the expected value of a binomial distribution?

Answer: $E(X) = np$

6 Hence write an expression for E(X) of a binomial distribution.

Answer: $\mathrm{E}(X) = \sum x\mathrm{P}(X=x) = np$

Developing inquiry skills

Consider now the second option from the original problem.

PAY \$20 and Toss 4 coins

If you get 4 Heads you win \$150

If you get 3 Heads you win your money back

If you get 2 Heads you win \$10

If you get 0 Heads or 1 Head you get nothing!

Is this a game that you think that if you played a few times you would end up with more than you started with?

Answer: Students use intuition to decide. Emphasise the large prize.

How could we tell?

Define a random variable for the winnings from playing the game once.

Answer: Let W be amount of winnings

Does this experiment fit a binomial distribution?

Yes – fixed number of trials (4, since you toss 4 coins); two possible outcomes (Heads = success, Tails = failure); constant probability of success (p=0.5); trials are independent.

What are the parameters?

Answer: n=4, p=0.5

What is the probability that you will win \$100? Your money back? \$10? \$0?

Answer: Let X represent the number of heads in 4 tosses of the coin. Then

$$P(X=0)=\left(\frac{1}{2}\right)^4=\frac{1}{16}$$

$$P(X=1)=4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)=\frac{4}{16}$$

$$\therefore \text{probability of \$0 is } \frac{1}{16}+\frac{4}{16}=\frac{5}{16}$$

$$P(X=2)=6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=\frac{6}{16}$$

$$\therefore \text{probability of \$10 is } \frac{6}{16}$$

$$P(X=3)=4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3=\frac{4}{16}$$

$$\therefore \text{probability of money back is } \frac{4}{16}$$

$$P(X=4)=\left(\frac{1}{2}\right)^4=\frac{1}{16}$$

$$\therefore \text{probability of \$100 is } \frac{1}{16}$$

Draw a probability distribution table for your random variable representing the winnings.

W	80	0	-10	-20
P(W=w)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{5}{16}$

Calculate the expected winnings from the game.

$$E(W)=\left(80\times\frac{1}{16}\right)+\left(0\times\frac{4}{16}\right)+\left(-10\times\frac{6}{16}\right)+\left(-20\times\frac{5}{16}\right)$$
$$=\frac{80}{16}+0-\frac{60}{16}-\frac{100}{16}$$
$$=-\frac{80}{16}=-5$$

This is a loss of \$5

How much would you expect to win or lose if you played the game 10 times? 100 times?

Answer: For 10 times, expected loss is \$50; for 100 times, expected loss is \$500.

Is the game 'fair'?

Answer: No, because $E(W)\neq 0$

What adjustment could you make to the prizes that would ensure that this is a fair game?

Answer: We want E(W) = 0

There are a number of ways to do this. For example, increase the prize for 4 heads to \$x. Then

$$E(W)=\left(x\times\frac{1}{16}\right)+\left(0\times\frac{4}{16}\right)+\left(-10\times\frac{6}{16}\right)+\left(-20\times\frac{5}{16}\right)=0$$
$$\frac{x}{16}-\frac{60}{16}-\frac{100}{16}=0$$
$$\frac{x}{16}-10=0$$
$$\frac{x}{16}=10$$
$$x=160$$

The first prize could be changed to \$180.

Other solutions and changes are possible

11.5 The normal distribution

Investigation 9

Conceptual understanding:

By visually representing the normal distribution the effects of changing the parameters μ and σ^2 can be observed.

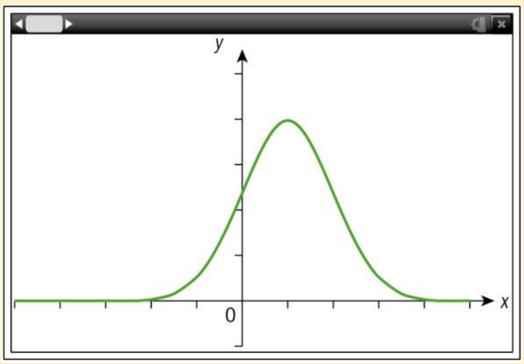
1 Use your GDC to graph the normal curves for each of the following normal variables:

A $X \sim \mathrm{N}(1,1)$ **B** $X \sim \mathrm{N}(2,1)$ **C** $X \sim \mathrm{N}(3,1)$ **D** $X \sim \mathrm{N}(4,1)$

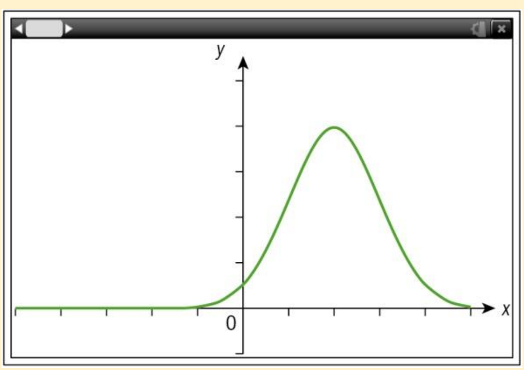
Describe differences and similarities between these normal curves. How does the value of the parameter μ affect the normal curve?

Answer:

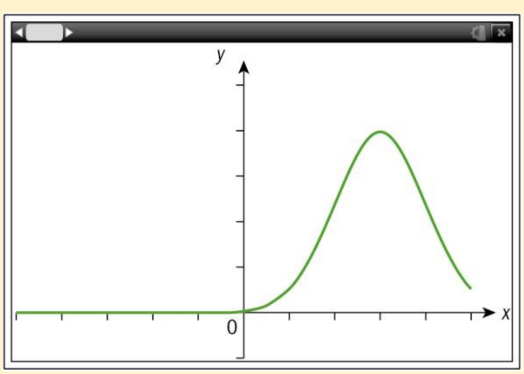
A



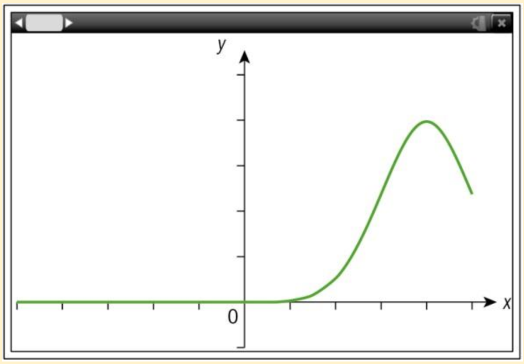
B



C



D

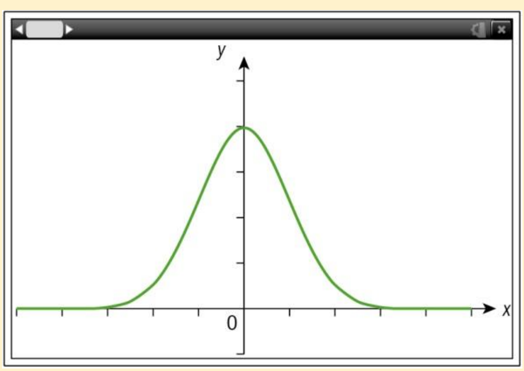


Differences	**Similarities**
The line of symmetry is dependent on the value of μ.	The curve is the same shape.
	The maximum value for y is the same.
	The domain and range are the same.

$x = \mu$ is the line of symmetry. Changing the value of μ shifts the graph left or right.

2 Graph the normal curves associated with these normal variables:

E



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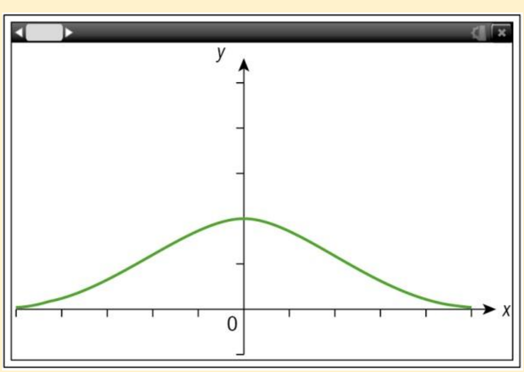

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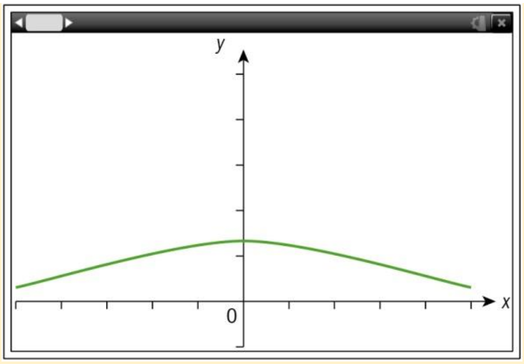

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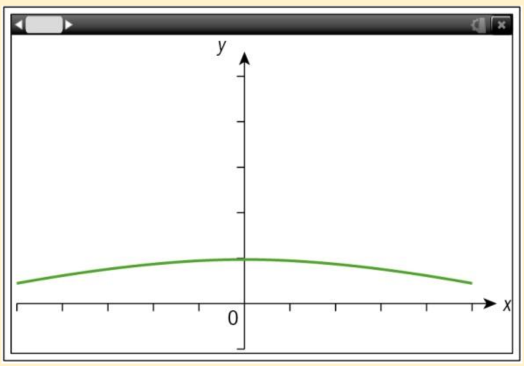

Describe differences and similarities between these normal curves. How does the value of the parameter σ^2 affect the normal curve?

Answer:

Differences	**Similarities**
The maximum value for y is different in each graph.	The line of symmetry is dependent on the value of $\boldsymbol{\mu}$ which is constant in this case.
The range is different.	The curve is the same shape.
	The domain is the same.

As σ^2 increases, the maximum value of y decreases. On the graph, changing the value of σ^2 changes the width of the distribution along the x-axis. Larger values of σ^2 produce distributions that are more spread out. The y-value is consequently lower, to ensure the sum of the probabilities equals 1.

3 Factual: Explore some other normal curves and write down your conclusions about the effects of the parameters on the graphs of the normal variables.

Answer: To generalize any normal distribution, two quantities have to be specified: the mean μ, where the peak of the density occurs, and the variance σ^2, which indicates the distribution of the bell curve.

4 Factual: The normal curves are defined by just two parameters: the mean and the variance of the distribution. Based on your knowledge of the shape of a normal curve, explain why it is not necessary to include the values of the median and the mode of the distribution.

Answer: As the curve is symmetrical, the mode and the median both take the same value as the mean. Therefore the only measure of central tendency required is the mean.

5 **Conceptual:** How is representing the normal distribution beneficial?

Answer: By visually representing the normal distribution the effects of changing the parameters μ and σ^2 can be observed.

TOK

Can you use mathematics to describe the world? Consider the use of probability methods in medical studies to assess risk factors for certain diseases.

Do we see mathematics in our world or are we imposing our own mathematical topics onto situations?

Developing inquiry skills

Consider now the second option from the original problem.

PAY \$20 and Toss 4 coins

If you get 4 Heads you win \$150

If you get 3 Heads you win your money back

If you get 2 Heads you win \$10

If you get 0 Heads or 1 Head you get nothing!

Is this a game that you think that if you played a few times you would end up with more than you started with?

Answer: Students use intuition to decide. Emphasise the large prize.

How could we tell?

Define a random variable for the winnings from playing the game once.

Answer: Let W be amount of winnings

Does this experiment fit a binomial distribution?

Yes – fixed number of trials (4, since you toss 4 coins); two possible outcomes (Heads = success, Tails = failure); constant probability of success (p=0.5); trials are independent.

What are the parameters?

Answer: n=4, p=0.5

What is the probability that you will win \$100? Your money back? \$10? \$0?

Answer: Let X represent the number of heads in 4 tosses of the coin. Then

$$P(X=0)=\left(\frac{1}{2}\right)^4=\frac{1}{16}$$

$$P(X=1)=4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)=\frac{4}{16}$$

$$\therefore \text{probability of \$0 is } \frac{1}{16}+\frac{4}{16}=\frac{5}{16}$$

$$P(X=2)=6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=\frac{6}{16}$$

$$\therefore \text{probability of \$10 is } \frac{6}{16}$$

$$P(X=3)=4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3=\frac{4}{16}$$

$$\therefore \text{probability of money back is } \frac{4}{16}$$

$$P(X=4)=\left(\frac{1}{2}\right)^4=\frac{1}{16}$$

$$\therefore \text{probability of \$100 is } \frac{1}{16}$$

Draw a probability distribution table for your random variable representing the winnings.

W	80	0	-10	-20
P(W=w)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{5}{16}$

Calculate the expected winnings from the game.

$$E(W)=\left(80\times\frac{1}{16}\right)+\left(0\times\frac{4}{16}\right)+\left(-10\times\frac{6}{16}\right)+\left(-20\times\frac{5}{16}\right)$$

$$=\frac{80}{16}+0-\frac{60}{16}-\frac{100}{16}$$

$$=-\frac{80}{16}=-5$$

This is a loss of \$5

How much would you expect to win or lose if you played the game 10 times? 100 times?

Answer: For 10 times, expected loss is \$50; for 100 times, expected loss is \$500.

Is the game 'fair'?

Answer: No, because $E(W) \neq 0$

What adjustment could you make to the prizes that would ensure that this is a fair game?

Answer: We want E(W) = 0

There are a number of ways to do this. For example, increase the prize for 4 heads to $*x*. Then

$$E(W) = \left(x \times \frac{1}{16}\right) + \left(0 \times \frac{4}{16}\right) + \left(-10 \times \frac{6}{16}\right) + \left(-20 \times \frac{5}{16}\right) = 0$$

$$\frac{x}{16} - \frac{60}{16} - \frac{100}{16} = 0$$

$$\frac{x}{16} - 10 = 0$$

$$\frac{x}{16} = 10$$

$$x = 160$$

The first prize could be changed to $180.

Other solutions and changes are possible

Random walking!

Approaches to Learning/learner profile: Critical thinking

Exploration Criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB Topic: Probability, Discrete Distributions

Introduction

This problem is designed to encourage students to think of simulation as a reasonable and acceptable approach to probability problems that may be too difficult to approach theoretically as they develop. This problem has a clearly stated aim and is accessible at first using Mathematical communication (Criterion B) and methods familiar to students from the chapter. The mathematics required to prove the result is quite difficult for some, but this should not restrict students from accessing it and using the tools available to them.

At the beginning of the problem students use basic coin tossing simulations and collect results as a class to produce more results and a hopefully more accurate answer. At the end of the task students are asked to consider using computer simulation. Coding for this is not only accessible for a computer science student or experienced programmer but can be learnt with a little effort and Personal engagement (Criterion C). Further personal engagement can be shown by extending the problem once the code has been mastered.

The proof of the result requires some understanding of probability distributions that is beyond the level of the course, but is a good opportunity for students to demonstrate more Personal engagement (Criterion C) and, if the proof is set out logically and clearly understood, could contribute to sophistication and rigour in the higher levels of Criterion E: Use of mathematics.

The problem

The problem is adapted from a famous problem in a branch of mathematical problems involving 'Random Walks'. Study of this branch has contributed to many different areas in physics and chemistry (Brownian motion and diffusion), biology (genetics, animal movements, population dynamics), economics (modelling share prices) and computer science (social media suggestions), amongst others.

Explore the problem

Since the man moves left or right with equal probability, a coin toss can be used to simulate this.

If appropriate, ask:

- *Why is a coin toss a suitable simulation?*

Students play the game 10 times and find the average number of steps taken. Discuss why this may not be an accurate result.

Ask:

- *What could you do to improve the accuracy of the average?*

Discuss the improved result based on a larger sample size.

The average may be getting closer, but you do not know if this is the actual number.

You can only be certain by proving the result theoretically.

Ask:

- *What have you noticed so far?*

(For example, always an odd number of steps, theoretically could go on forever, etc.)

Calculate probabilities

Students may need help when constructing the tree diagram.

Ask:

- *What are the limitations of using a tree diagram in this case?*

The tree diagram is very large! It becomes impossible to draw after 6 or 7 tosses.

Remind students how to use a tree diagram to find probabilities.

If needed, to help students find the probability that the man falls into the ditch after a total of exactly 5 steps, ask:

- *If the man moves Left (L) then Left again (L) and then Right (R) and then Left (L) and then Left (L) then he will be in the ditch. Where is this scenario on your tree diagram?*
- *What is the probability that the man takes this particular sequence of steps? In other words what is the probability of TTHTT?*

Probability is $(0.5)^5 = 0.03125 = \frac{1}{32}$

- *What other sequences of coin tosses will lead to the man falling into the ditch after* exactly *5 steps?*

- *TTHTT, THTTT, THHHH, HTTTT, HTHHH, HHTHH*
- *What are the probabilities associated with each of these sequences?*

They are all $(0.5)^5 = 0.03125 = \frac{1}{32}$

- *What is the probability that the man falls into the ditch after a total of exactly 5 steps?*

$6(0.5)^5 = 6(0.03125) = \frac{3}{16}$

Minimum number of steps to fall into the ditch is 3.

Maximum is infinite.

Probability that the man falls into the ditch after a total of exactly 3 steps is

$2(0.5)^3 = 2(0.125) = \frac{1}{4}$

To explain why all the paths have an odd number of steps:

It is an odd number of steps because:

From the centre, after the first step, the man will always be one step away from the centre (two steps away from the ditch on that side).

From here, after 2 steps, he will either be in the ditch on the same side, back to the same position, or one step from the centre (two steps away from the ditch) on the other side. This will repeat.

This gives 1 + a multiple of 2 which is odd.

You could use a diagram to demonstrate this, with the centre shown by a black dot, one step from the centre on either side shown by a red dot, two steps away from the centre on each side shown by a black dot, and the ditches shown by blue dots.

The probabilities are:

x	1	2	3	4	5	6	7	8	9	10	11	12	....
P(*X* = *x*)	0	0	$\frac{1}{4}$	0	$\frac{3}{16}$	0	$\frac{91}{64}$	0	$\frac{27}{256}$	0	$\frac{81}{1024}$	0	

The table alternates between 0 and a value.

The next value is calculated by multiplying by $\frac{3}{4}$.

Ask:

- *Can you explain where the value of $\frac{3}{4}$ comes from in this situation?*

Again, you could use diagrams to demonstrate this.

Simulation

Students could use the formula $E(X)=\sum_{1}^{\infty} xP(X=x)$ to try to find E(X), the expected number of steps that would be required.

This will give them the exact theoretical answer to the problem posed.

Note: This is beyond the level of the course and may require students to investigate Arithmetico–Geometric series.

This is a complicated problem because there are an infinite number of values of x that will result in ending in the ditch. It is *possible* to calculate the expected value but it requires mathematics that will be beyond both the SL and HL syllabus.

However, some HL students will be able to give it a go – perhaps with some scaffolding. It can be emphasized that, in doing so, (if this were an exploration) this could improve the level of Personal engagement (Criterion C) and if the mathematics is presented well and is clearly understood, then it could be considered sophisticated and hence reach the higher level in Criterion E: Use of mathematics.

Here is the solution written out:

$$\mathrm{E}(X)=\sum_{x=1}^{\infty} x\times \mathrm{P}(X=x)$$

$$=(1\times 0)+(2\times 0)+\left(3\times \tfrac{1}{4}\right)+(4\times 0)+\left(5\times \tfrac{3}{16}\right)+(6\times 0)+\left(7\times \tfrac{9}{64}\right)+\cdots$$

$$=\left(3\times \frac{1}{4}\right)+\left(5\times \frac{3}{16}\right)+\left(7\times \frac{9}{64}\right)+\left(9\times \frac{27}{264}\right)+\cdots$$

$$=\frac{1}{4}\left(3+5\times \frac{3}{4}+7\times \frac{9}{16}+9\times \frac{27}{64}+\cdots\right)$$

$$=\frac{1}{4}(3+5\times \frac{3}{4}+7\times \left(\frac{3}{4}\right)^{3}+\cdots)$$

This is an infinite arithmetico-geometric series, its sum can be found neatly as follows:

$$\mathrm{E}(X)=\tfrac{1}{4}[3+2\times \left(\tfrac{3}{4}\right)+2\times \left(\tfrac{3}{4}\right)^{2}+2\times \left(\tfrac{3}{4}\right)^{3}+\cdots$$

$$\tfrac{3}{4}\times \mathrm{E}(X)=\tfrac{1}{4}[3\times \left(\tfrac{3}{4}\right)+5\times \left(\tfrac{3}{4}\right)^{2}+7\times \left(\tfrac{3}{4}\right)^{3}+9\times \left(\tfrac{3}{4}\right)^{4}+\cdots$$

Subtracting,

$$\frac{1}{4}\mathrm{E}(X) = \frac{1}{4}[3 + 2 \times \left(\frac{3}{4}\right) + 2 \times \left(\frac{3}{4}\right)^2 + 2 \times \left(\frac{3}{4}\right)^3 + \cdots]$$

$$\mathrm{E}(X) = 3 + 2 \times \left(\frac{3}{4}\right) + 2 \times \left(\frac{3}{4}\right)^2 + 2 \times \left(\frac{3}{4}\right)^3 + \cdots$$

$$\mathrm{E}(X) = 3 + 2 \times \frac{3}{4} \times \left[1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots\right]$$

$$= 3 + 2 \times \frac{3}{4} \times \frac{1}{1-\frac{3}{4}}$$

$$= 3 + 2 \times \frac{3}{4} \times 4$$

$$= 9$$

This shows that the theoretical result is 9. Compare this with the result from the earlier simulation by the class.

At this stage, you could tell the class that the expected number of steps is 9.

Ask:

- *Why do you think simulations are used?*

Collecting and recording large numbers of results by hand is very time consuming and can be very expensive.

Here are some examples of computer coding that could be copied or shown to the students. It would be possible to replicate these results on most computer coding systems. (If there are computer science students in the class then they could be encouraged to help the class, although simple coding should be accessible to all!)

Note: The idea here is that *simple* computer coding is accessible to all students with a little work, and then more complicated problems can be solved. This is considerably more efficient than more manual methods. It is also worth pointing out that this is actually an exceptionally real process in many, many fields of work such as meteorology, disaster management, economics and finance, sports predictions, etc. This is one of the real avenues of work for the modern mathematician!

Extension

Suggestions of how students could vary the problem:

- Change the start position.
- Use a bias coin.
- Change the number of steps from the centre to the ditch.
- Change the problem to 2 dimensions.

Students may also be able to devise their own probability question which they could answer using simulation.